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Curved 11D supergeometry

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Abstract: We examine the theta-expansion of the eleven-dimensional supervielbein. We outline a systematic procedure which can be iterated to any order. We give explicit expressions for the vielbein and three-form potential components up to order $\mathcal{O}(\theta^5)$. Furthermore we show that at each order in the number of supergravity fields, in a perturbative expansion around flat space, it is possible to obtain exact expressions to all orders in theta. We give the explicit expression at linear order in the number of fields and we show how the procedure can be iterated to any desired order. As a byproduct we obtain the complete linear coupling of the supermembrane to the background supergravity fields, covariantly in component form. We discuss the implications of our results for M(atrix) theory.

Keywords: M-Theory, M(atrix) Theories, Superspaces.

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Contents

1. Introduction

It has been hoped that the quantization of the world-volume theory of the eleven-dimensional supermembrane [1, 2], in analogy to the quantization of the superstring, would furnish a microscopic description of M-theory. However, a head-on approach to this problem has been stalled by certain features of the membrane which make it much less tractable than the superstring: the presence of nonlinearities and the absence of conformal invariance.

On the other hand, it has been known for some time that supersymmetric matrix quantum mechanics (henceforth: matrix model) emerges as the finite-N regularization of the world-volume theory of the supermembrane in flat eleven-dimensional spacetime in the light-cone frame [3]. Excitement at the prospect of being able to describe the supermembrane by such a simple theory as ordinary quantum mechanics was temporarily halted by the realization that the spectrum of the theory was continuous [4]. The latter fact was initially thought to signal the instability of the membrane.

More recently, however, the large-N limit of the matrix model was conjectured to give the complete description of M-theory in uncompactified flat space in the "infinite momentum frame" [5], and the issue of the continuous spectrum was resolved by recognizing that the Hilbert space of the theory contains multi-particle states. The original BFSS conjecture[1] was later extended to finite N [6]. According to this, finite-N supersymmetric quantum mechanics describes the discretized light-cone quantization of M-theory with N units of compact momentum. It was subsequently argued that the finite-N conjecture follows from the BFSS [7, 8].

The argument of [7, 8] is still valid in the case of toroidal compactifications of M-theory. An extension of the conjecture to include general curved backgrounds is much less clear however, and several complicating issues arise [9, 10]. The original BFSS conjecture was recently generalized in [11] to include M-theory on the maximally-supersymmetric eleven-dimensional plane wave background. In [12] the linear couplings of the matrix model to general background supergravity fields were derived within the context of matrix theory, up to quadratic order in the fermionic membrane coordinates (θ).

The same problem was approached from the point of view of the supermembrane in [13], generalizing the work of [14] for the superparticle. The authors of [13] constructed the light-cone gauge-fixed membrane vertex operators describing the interactions of massless supermultiplets in the linearized approximation. These vertex operators are nothing but the linear couplings of the membrane to a general supergravity background. Contact with the matrix model was made by subsequently performing a finite-N regularization. A covariant computation to all orders in θ was deemed unfeasible and was not attempted. In fact no covariant vertex operators had been constructed, even for the superstring or the superparticle, until recently within Berkovits' pure-spinor approach [15, 16]. The vertex operators can be used to compute (or rather: to define) membrane scattering amplitudes [17, 18]. They are therefore interesting in their own right, in the hope that they may prove useful in probing the dynamics of M-theory.

The (component form of the) world-volume theory of the supermembrane in a general supergravity background is also an essential input in membrane-instanton computations [19]–[23]. Such nonperturbative effects have recently attracted a lot of attention in the context of cosmological models with moving branes. (The literature on the subject is already vast: see for example [24]–[26] and papers citing those).

In principle, all information about the coupling of the eleven-dimensional supermembrane to the background supergravity fields is contained in the action of [1, 2]. The latter, however, is given in superspace coordinates, in terms of the background supervielbein. Extracting information in component form from the supermembrane action boils down to

[1]Somewhat confusing is the fact that the BFSS succeeded the so-called "old matrix model", but the latter has recently succeeded in re-emerging as the new new one; as a result the BFSS is now the new old matrix model.

the problem of obtaining the θ-expansion of the supervielbein. For a general superspace geometry, this problem has so far only been tackled iteratively. Clearly, iterating all the way up to θ^{32} (when the series terminates) is exceedingly tedious. All-order results are only known in special cases, namely for flat space and for the coset superspaces $AdS_4 \times S^7$, $AdS_7 \times S^4$ [27]–[30] (for a related type IIB discussion of $AdS_5 \times S^5$ see [31]).

Partial results regarding terms to order θ^2 in the expansion of the eleven-dimensional supervielbein have appeared in the literature in [32]. The authors of [32] used a method called "gauge completion" which, as far as elegance is concerned, leaves a lot to be desired. The authors of [33] computed explicitly up to $\mathcal{O}(\theta^3)$ for a general background (and up to $\mathcal{O}(\theta^4)$ for a bosonic background) using the method of normal coordinates [34]. The latter is also known as the "covariant θ-expansion" — the terminology referring to the fact that the gauge fields (graviton, gravitino) enter through the supervielbein and superconnection, and their derivatives enter through covariant field-strengths. The method of normal coordinates takes advantage of the superspace formulation [35, 36] of eleven-dimensional supergravity [37] and, as opposed to gauge completion, is unambiguous and systematic.

In this paper we use normal coordinates (we take the point of view that it is the most natural generalization of the Wess-Zumino gauge) to derive recursion relations between different levels of the θ-expansion. All results are eventually expressed in component form. We obtain the explicit expansion of the supervielbein to order $\mathcal{O}(\theta^5)$ and we find agreement with the existing literature referred to in the previous paragraph. More significantly, however, our way of presenting the recursion relations makes possible the following important observation: at each order in the number of fields, in a perturbative expansion around flat space, *it is possible to obtain exact expressions to all orders in θ*. We derive the explicit expressions at linear order in the number of fields, in section 6.1. We also explain how the procedure can be iterated to higher orders.

Let us sketch the basic idea of our method. Our particular gauge-fixing choice leads to the equations[2]

$$\begin{aligned}\theta^\mu (E_\mu{}^A - \delta_\mu{}^A) &= 0\\ \theta^\mu \Omega_{\mu A}{}^B &= 0\end{aligned}$$

and

$$\theta^\mu \partial_\mu = \theta^\mu \delta_\mu^\alpha \nabla_\alpha \,.$$

The above allow us to systematically translate the order of the θ-expansion to the number of spinor derivatives, where the action of the latter on the various superfields is known. We note that knowledge of the θ-expansion of the superfield $T_{ab}{}^\alpha$, the covariant gravitino field-strength, suffices to obtain the θ-expansion of all other superfields, the vielbein in particular. On the other hand, in an expansion around flat space, the n-th level $(T^{(n)}{}_{ab}{}^\alpha)$ of the θ-expansion of the gravitino field-strength can be written schematically

[2]We employ standard superspace notation. Further details can be found in the following sections.

as

$$
\begin{aligned}
T^{(n)} &\sim \frac{\mathcal{O}^{\frac{n}{2}}}{n!}\partial\Psi + U^{(n)}\,, \qquad n = 2k\,,\\
T^{(n)} &\sim \frac{\mathcal{O}^{\frac{n-1}{2}}}{n!}(\theta R + \theta\partial G) + U^{(n)}\,, \qquad n = 2k+1\,,
\end{aligned}
$$

where $U^{(n)}$ is a known expression nonlinear in the fields and $\mathcal{O}$ is a (matrix) differential operator quadratic in θ; schematically, $\mathcal{O} \sim (\theta\Gamma\theta)\partial$. We have denoted by Ψ, R, G, the gravitino, Riemann tensor and four-form field strength of eleven-dimensional supergravity, respectively. Since $U^{(n)}$ is nonlinear, the equations above can be iterated to any order in the number of fields. In other words: *perturbatively in the number of fields we can obtain expressions which are exact to all orders in θ.*

In the following section 2 we review the superspace formulation of eleven-dimensional supergravity. In section 3 we describe the gauge-fixing procedure. In section 4 we obtain recursion relations which are then iterated in section 5 to obtain the explicit expansion of the vielbein and three-form potential to order $\mathcal{O}(\theta^5)$. The expansion in the number of fields is described in section 6. The linear coupling of the covariant supermembrane to the background fields, is given in section 7. Section 8 contains a discussion of future directions and possible applications of our results. Since this is a somewhat technical paper, for quick reference we have included an index of various definitions used, in appendix A. In appendix B we have included a note on our conventions concerning gamma matrices and spinor notation. Appendix C contains the vielbein and three-form potential expansions for the simplified case of a purely geometric background.

2. On-shell 11D supergravity in superspace

This section is a summary of known results that can be found in or deduced from the literature. We have included it in order to establish notation and conventions, and to make the paper self-contained.

Eleven-dimensional supergravity [37] admits a superspace formulation [35, 36]. Let $A = (a,\alpha)$; $a = 0\ldots 10$, $\alpha = 1\ldots 32$, be a flat superspace index and let $E^A = (E^a, E^\alpha)$ be the coframes of the $(11|32)$ supermanifold. Moreover, let us introduce a connection one-form $\Omega_A{}^B$ with lorentzian structure group. The supertorsion and supercurvature are given by

$$
\begin{aligned}
T^A &= DE^A := dE^A + E^B\Omega_B{}^A = \frac{1}{2}E^C E^B T_{BC}{}^A\\
R_A{}^B &= d\Omega_A{}^B + \Omega_A{}^C\Omega_C{}^B = \frac{1}{2}E^D E^C R_{CD,A}{}^B
\end{aligned}
\tag{2.1}
$$

and obey the Bianchi identities

$$
\begin{aligned}
DT^A &= E^B R_B{}^A\\
DR_B{}^A &= 0\,.
\end{aligned}
\tag{2.2}
$$

Note that for a lorentzian structure group the second Bianchi identity follows from the first [38]. In a purely geometrical definition in terms of the supertorsion, it was shown in [39] that the equations of motion of 11D supergravity follow from the constraint

$$T_{\alpha\beta}{}^{a} = -i(\Gamma^{a})_{\alpha\beta}\,. \tag{2.3}$$

In this formulation the physical fields of the theory, the graviton, the gravitino and the three-form potential, appear through their covariant field strengths. Namely, the curvature $R_{ab,c}{}^{d}$ is identified with the top component of the supercurvature, the gravitino field-strength $T_{ab}{}^{\alpha}$ is identified with the dimension three-halves component of the supertorsion, while the four-form field strength G_{abcd} appears in the dimension-one components of the supertorsion and supercurvature.

More explicitly, the remaining nonzero components of the supertorsion and supercurvature of undeformed 11D supergravity are given by

$$T_{a\beta}{}^{\gamma} = (\mathcal{T}_{a}{}^{bcde})_{\beta}{}^{\gamma} G_{bcde} \tag{2.4}$$

and

$$\begin{aligned} R_{\alpha\beta,ab} &= i(\mathcal{R}_{ab}{}^{cdef})_{\alpha\beta} G_{cdef} \\ R_{\alpha b,cd} &= i(\mathcal{S}_{bcd}{}^{ef})_{\alpha\beta} T_{ef}{}^{\beta}\,, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \mathcal{T}_{a}{}^{bcde} &:= -\frac{1}{36}\left(\delta_{a}^{[b}\Gamma^{cde]} + \frac{1}{8}\Gamma_{a}{}^{bcde}\right) \\ \mathcal{R}_{ab}{}^{cdef} &:= \frac{1}{6}\left(\delta_{a}^{[c}\delta_{b}^{d}\Gamma^{ef]} + \frac{1}{24}\Gamma_{ab}{}^{cdef}\right) \\ \mathcal{S}_{bcd}{}^{ef} &:= \frac{1}{2}\left(\Gamma_{b}\delta_{c}^{[e}\delta_{d}^{f]} + \Gamma_{c}\delta_{b}^{[e}\delta_{d}^{f]} - \Gamma_{d}\delta_{b}^{[e}\delta_{c}^{f]}\right). \end{aligned} \tag{2.6}$$

Note that the Lorentz condition implies

$$R_{AB\alpha}{}^{\beta} = \frac{1}{4} R_{ABcd}(\Gamma^{cd})_{\alpha}{}^{\beta}\,. \tag{2.7}$$

The action of the spinorial derivative on the physical field strengths is given by

$$\begin{aligned} \nabla_{\alpha} G_{abcd} &= 6i(\Gamma_{[ab}T_{cd]})_{\alpha} \\ \nabla_{\alpha} T_{ab}{}^{\beta} &= \frac{1}{4} R_{ab,cd}(\Gamma^{cd})_{\alpha}{}^{\beta} - 2\nabla_{[a}T_{b]\alpha}{}^{\beta} - 2T_{[a|\alpha}{}^{\epsilon}T_{|b]\epsilon}{}^{\beta} \\ \nabla_{\alpha} R_{ab,cd} &= 2\nabla_{[a|}R_{\alpha|b]cd} - T_{ab}{}^{\epsilon}R_{\epsilon\alpha cd} + 2T_{[a|\alpha}{}^{\epsilon}R_{\epsilon|b]cd}\,. \end{aligned} \tag{2.8}$$

The equations-of-motion that follow from (2.8) read

$$\begin{aligned} \nabla_{[a}G_{bcde]} &= 0 \\ \nabla^{f} G_{fabc} &= -\frac{1}{2(4!)^{2}}\varepsilon_{abcd_1\ldots d_8} G^{d_1\ldots d_4} G^{d_5\ldots d_8} \end{aligned}$$

$$
\begin{aligned}
(\Gamma^a T_{ab})_\alpha &= 0 \\
R_{ab} - \frac{1}{2}\eta_{ab} R &= -\frac{1}{12}\left(G_{adfg} G_b{}^{dfg} - \frac{1}{8}\eta_{ab} G_{dfge} G^{dfge} \right) .
\end{aligned} \tag{2.9}
$$

The equations above imply the existence of a closed superfour-form G which can be written locally in terms of a superpotential C,

$$
G = dC \,. \tag{2.10}
$$

The only nonvanishing components of G are

$$
G_{abcd}\,, \qquad G_{\alpha\beta ab} = -i(\Gamma_{ab})_{\alpha\beta}\,. \tag{2.11}
$$

In order to derive the equations of motion in component form, a little more work is required. Equations (2.10), (2.11) together with

$$
G_{MNPQ} = (-)^{m(c+n+b+p+a+q)}(-)^{n(b+p+a+q)}(-)^{p(a+q)} E_Q{}^A E_P{}^B E_N{}^C E_M{}^D G_{DCBA} \tag{2.12}
$$

imply

$$
\boxed{e_m{}^a e_n{}^b e_p{}^c e_q{}^d G^{(0)}_{abcd} = 4\partial_{[m} C^{(0)}_{npq]} - 6i(\Psi_{[m}\Gamma_{np}\Psi_{q]})\,,} \tag{2.13}
$$

where for any superfield S,

$$
S^{(0)} := S|_{\theta=0} \tag{2.14}
$$

and we define

$$
\begin{aligned}
e_m{}^a &:= E^{(0)}_m{}^a \\
\Psi_m{}^\alpha &:= E^{(0)}_m{}^\alpha \\
\omega_{mA}{}^B &:= \Omega^{(0)}_{mA}{}^B \,.
\end{aligned} \tag{2.15}
$$

It follows from definition (2.1) that

$$
T^{(0)}_{mnk} = 2(\partial_{[m} e_{n]}{}^a e_{ak} + \omega_{[mn]k})\,, \tag{2.16}
$$

where we have defined

$$
\begin{aligned}
T^{(0)}_{mnk} &:= T^{(0)}_{mn}{}^a e_{ak} \\
\omega_{mnk} &:= \omega_{ma}{}^b e_n{}^a e_{bk}\,.
\end{aligned} \tag{2.17}
$$

On the other hand,

$$
T_{MN}{}^A = (-)^{m(n+c)} E_N{}^C E_M{}^B T_{BC}{}^A \tag{2.18}
$$

together with (2.3) implies

$$
T^{(0)}_{mn}{}^a = i(\Psi_m \Gamma^a \Psi_n)\,, \tag{2.19}
$$

where we have suppressed spinor indices for simplicity. Combining (2.16), (2.19) we arrive at

$$
\omega_{nkm} = \overset{0}{\omega}_{nkm} + K_{nkm}\,, \tag{2.20}
$$

where

$$\begin{aligned}
\overset{0}{\omega}_{nkm} &:= \partial_{[k}e_{m]}{}^{a}e_{an} - \partial_{[m}e_{n]}{}^{a}e_{ak} - \partial_{[n}e_{k]}{}^{a}e_{am} \\
K_{nkm} &:= \frac{i}{2}(\Psi_m\Gamma_k\Psi_n + \Psi_n\Gamma_m\Psi_k - \Psi_k\Gamma_n\Psi_m)
\end{aligned} \tag{2.21}$$

and

$$\Gamma_m := e_m{}^{a}\Gamma_a\,. \tag{2.22}$$

Note that the contorsion tensor K satisfies

$$K_{[mn]p} = \frac{1}{2}T^{(0)}_{mnp}\,. \tag{2.23}$$

With $\overset{0}{\omega}$ we can associate a covariant derivative $\overset{0}{\mathcal{D}}$ which obeys

$$\overset{0}{\mathcal{D}}_n\, e_m{}^{a} := \partial_n e_m{}^{a} - \overset{0}{\Gamma}{}^{k}_{nm} e_k{}^{a} + \overset{0}{\omega}_{nb}{}^{a} e_m{}^{b} = 0\,, \tag{2.24}$$

where $\overset{0}{\Gamma}$ is the Levi-Civita connection. Similarly, with ω we can associate a covariant derivative $\mathcal{D}$ which obeys

$$\mathcal{D}_n e_m{}^{a} := \partial_n e_m{}^{a} - \Gamma^{k}_{nm} e_k{}^{a} + \omega_{nb}{}^{a} e_m{}^{b} = 0\,, \tag{2.25}$$

where

$$\Gamma^{p}_{mn} = \overset{0}{\Gamma}{}^{p}_{mn} + K_{mn}{}^{p}\,. \tag{2.26}$$

Proceeding as above, one can show that

$$\boxed{e_m{}^{a}e_n{}^{b}\, T^{(0)}_{ab}{}^{\alpha} = \partial_m\Psi_n{}^{\alpha} + \omega_{m\beta}{}^{\alpha}\Psi_n{}^{\beta} + (\Psi_m\mathcal{T}_n{}^{abcd})^{\alpha}G^{(0)}_{abcd} - (m \leftrightarrow n)\,,} \tag{2.27}$$

where

$$\mathcal{T}_n{}^{abcd} := e_n{}^{f}\mathcal{T}_f{}^{abcd}\,. \tag{2.28}$$

Finally, (2.1), (2.19) together with

$$R_{MNA}{}^{B} = (-)^{m(n+d)}E_N{}^{D}E_M{}^{C}R_{CDA}{}^{B} \tag{2.29}$$

imply that

$$\boxed{e_m{}^{a}e_n{}^{b}e_k{}^{c}e_l{}^{d}R^{(0)}_{abcd} = R(\omega)_{mnkl} + i(\Psi_m\mathcal{R}_{kl}{}^{abcd}\Psi_n)G^{(0)}_{abcd} - 2i(\Psi_{[m}\mathcal{S}_{n]kl}{}^{ab}T^{(0)}_{ab})\,,} \tag{2.30}$$

where

$$\begin{aligned}
\mathcal{R}_{kl}{}^{cdfg} &:= e_k{}^{a}e_l{}^{b}\mathcal{R}_{ab}{}^{cdfg} \\
\mathcal{S}_{nkl}{}^{cd} &:= e_n{}^{f}e_k{}^{a}e_l{}^{b}S_{fab}{}^{cd}.
\end{aligned} \tag{2.31}$$

We arrive at the equations of motion in component form by taking (2.9) at $\theta = 0$ and substituting (2.13), (2.27), (2.30).

3. Gauge-fixing

The supervielbein contains an enormous amount of gauge freedom which can be fixed by using higher θ-levels of supersymmetry and Lorentz transformations. In this section we impose the most natural generalization, to all levels of the θ-expansion, of the Wess-Zumino gauge. This leads to the system of normal coordinates introduced in [34] as a superspace generalization of the ordinary normal-coordinate expansion on riemannian manifolds. As we will see below, significant simplifications occur when superspace quantities are expressed in these coordinates.

Let us first introduce some notation. For any superfield $S_{\{A\}}$ we define the coefficients in the θ-expansion as follows:

$$S_{\{A\}} = \sum_{n=0}^{32} S^{(n)}_{\{A\}}\,, \tag{3.1}$$

where

$$S^{(n)}_{\{A\}} := \frac{1}{n!}\theta^{\mu_n}\dots\theta^{\mu_1} S^{(n)}_{\mu_1\dots\mu_n,\{A\}} \tag{3.2}$$

and $\{A\}$ stands for all Lorentz indices. In particular, the θ-expansions of the vielbein and superconnection read

$$\begin{aligned} E_M{}^A &= \sum_{n=0}^{32} E^{(n)}_M{}^A \\ &= \sum_{n=0}^{32} \frac{1}{n!}\theta^{\mu_n}\dots\theta^{\mu_1} E^{(n)}_{\mu_1\dots\mu_n,M}{}^A \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \Omega_{MA}{}^B &= \sum_{n=0}^{32} \Omega^{(n)}_{MA}{}^B \\ &= \sum_{n=0}^{32} \frac{1}{n!}\theta^{\mu_n}\dots\theta^{\mu_1} \Omega^{(n)}_{\mu_1\dots\mu_n,MA}{}^B\,, \end{aligned} \tag{3.4}$$

respectively. Under supersymmetry and Lorentz tranformations with parameters ξ^A and $L_A{}^B$ respectively, the vielbein and superconnection transform as [40]

$$\delta E_M{}^A = -\nabla_M\xi^A - \xi^B T_{BM}{}^A + E_M{}^B L_B{}^A \tag{3.5}$$

and

$$\delta\Omega_{MA}{}^B = -\xi^C R_{MCA}{}^B + \Omega_{MA}{}^C L_C{}^B - \Omega_{MC}{}^B L_A{}^C - \partial_M L_A{}^B\,, \tag{3.6}$$

where

$$\nabla_M\xi^A := \partial_M\xi^A + \Omega_{MB}{}^A\xi^B\,. \tag{3.7}$$

As can be seen from (3.5), we can use $\xi^{(1)}_{\mu,}{}^A$ to set

$$\begin{aligned} E^{(0)}_\mu{}^a &= 0 \\ E^{(0)}_\mu{}^\alpha &= \delta^\alpha_\mu\,. \end{aligned} \tag{3.8}$$

More generally, we can use $\xi^{(n+1)}_{\mu_1\dots\mu_{n+1},}{}^{A}$ to set to zero the totally antisymmetric parts

$$E^{(n)}_{[\mu_1\dots\mu_n,\mu]}{}^{A} = 0\,, \qquad 1 \leq n \leq 32\,. \tag{3.9}$$

Similarly, from (3.6) we see that we can use $L^{(1)}_{\mu,A}{}^{B}$ to set

$$\Omega^{(0)}{}_{\mu A}{}^{B} = 0\,. \tag{3.10}$$

The higher θ-levels of the Lorentz transformation, $L^{(n+1)}_{\mu_1\dots\mu_{n+1},A}{}^{B}$, can be used to set

$$\Omega^{(n)}_{[\mu_1\dots\mu_n,\mu]A}{}^{B} = 0\,, \qquad 1 \leq n \leq 32\,. \tag{3.11}$$

The gauge-fixing conditions (3.8)–(3.11) imply

$$\begin{aligned} \theta^\mu E_\mu{}^a &= 0 \\ \theta^\mu E_\mu{}^\alpha &= \theta^\alpha \end{aligned} \tag{3.12}$$

and

$$\theta^\mu \Omega_{\mu A}{}^{B} = 0\,, \tag{3.13}$$

where we have defined

$$\theta^\alpha := \theta^\mu \delta^\alpha_\mu\,. \tag{3.14}$$

Expressions (3.12), (3.13) are identical to [34, (A.3),(A.4)].

The inverse $E_A{}^M$ of the vielbein satisfies

$$E_A{}^M E_M{}^B = \delta^B_A\,. \tag{3.15}$$

At zeroth-order in the θ-expansion we have

$$\begin{aligned} E^{(0)}_{\alpha}{}^{m} &= 0 \\ E^{(0)}_{\alpha}{}^{\mu} &= \delta^\mu_\alpha \\ E^{(0)}_{a}{}^{\mu} &= -\Psi_a{}^\mu := -e_a{}^m \Psi_m{}^\alpha \delta^\mu_\alpha \\ E^{(0)}_{a}{}^{m} &= e_a{}^m\,. \end{aligned} \tag{3.16}$$

At order $n \geq 1$ we get

$$\begin{aligned} E^{(n)}_{A}{}^{m} &= -\sum_{r=0}^{n-1} E^{(r)}_{A}{}^{M} E^{(n-r)}_{M}{}^{a} e_a{}^m \\ E^{(n)}_{A}{}^{\mu} &= \sum_{r=0}^{n-1} \left(-E^{(r)}_{A}{}^{M} E^{(n-r)}_{M}{}^{\alpha} \delta^\mu_\alpha + E^{(r)}_{A}{}^{M} E^{(n-r)}_{M}{}^{a} \Psi_a{}^\mu \right)\,. \end{aligned} \tag{3.17}$$

Using (3.16), (3.17), (3.12) it is straightforward to prove by induction that

$$\begin{aligned} \theta^\alpha E_\alpha{}^m &= 0 \\ \theta^\alpha E_\alpha{}^\mu &= \theta^\mu\,. \end{aligned} \tag{3.18}$$

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Moreover, from (3.18), (3.13) it follows that

$$\theta^\alpha \Omega_{\alpha A}{}^B = 0\,. \tag{3.19}$$

We also have that for any superfield S (suppressing any Lorentz indices)

$$\begin{aligned}\theta^\alpha \nabla_\alpha S &= \theta^\alpha E_\alpha{}^M(\partial_M + \Omega_M)S\\ &= \theta^\mu \partial_\mu S\end{aligned} \tag{3.20}$$

and therefore

$$S^{(n)} = \frac{(n-r)!}{n!}\,\theta^{\alpha_r}\ldots\theta^{\alpha_1}\,(\nabla_{\alpha_1}\ldots\nabla_{\alpha_r}S)^{(n-r)}\,. \tag{3.21}$$

Equation (3.20) is the same as[34, eq. (1)].

4. Recursion

4.1 G_{abcd},$T_{ab}{}^\alpha$ and R_{abcd}

Using the results in the previous sections we can now obtain recursion relations that relate different levels in the θ-expansions of the various superfields. First let us first derive the recursion relations for G, T, R. These are readily obtained from (3.21), (2.8):

$$\begin{aligned}G^{(n)}_{abcd} &= \frac{6i}{n}(\theta\Gamma_{[ab}T^{(n-1)}_{cd]})\\ T^{(n)}_{ab}{}^{\alpha} &= \frac{1}{4n}(\theta\Gamma^{cd})^\alpha R^{(n-1)}_{abcd} + \frac{2}{n}(\theta\mathcal{T}_{[a}{}^{cdef})^\alpha(\nabla_{b]}G_{cdef})^{(n-1)} -\\ &\quad -\frac{2}{n}(\theta\mathcal{T}_{[a}{}^{cdef}\mathcal{T}_{b]}{}^{c'd'e'f'})^\alpha(G_{cdef}G_{c'd'e'f'})^{(n-1)}\\ R^{(n)}_{abcd} &= -\frac{2i}{n}(\theta\mathcal{S}_{[a|cd}{}^{ef})_\alpha(\nabla_{|b]}T_{ef}{}^\alpha)^{(n-1)} - \frac{i}{n}(\theta\mathcal{R}_{cd}{}^{efgh})_\alpha(T_{ab}{}^\alpha G_{efgh})^{(n-1)} +\\ &\quad +\frac{2i}{n}(\theta\mathcal{T}_{[a}{}^{efgh}\mathcal{S}_{b]cd}{}^{e'f'})_\alpha(T_{e'f'}{}^\alpha G_{efgh})^{(n-1)}\,.\end{aligned} \tag{4.1}$$

For later use let us also note that

$$T^{(n)}_{ab}{}^{\alpha} = \frac{i}{n(n-1)}\{(\mathcal{M}_{[a|}{}^{ef})^\alpha{}_\beta(\nabla_{|b]}T_{ef}{}^\beta)^{(n-2)} + (\mathcal{N}_{ab}{}^{c_1\ldots c_6})^\alpha{}_\beta(G_{c_1\ldots c_4}T_{c_5c_6}{}^\beta)^{(n-2)}\}\,, \tag{4.2}$$

where

$$\begin{aligned}(\mathcal{M}_a{}^{ef})^\alpha{}_\beta &:= -\frac{1}{2}(\theta\Gamma^{bc})^\alpha(\theta\mathcal{S}_{abc}{}^{ef})_\beta + 12(\theta\mathcal{T}_a{}^{bcef})^\alpha(\theta\Gamma_{bc})_\beta\\ (\mathcal{N}_{ab}{}^{c_1\ldots c_6})^\alpha{}_\beta &:= -\frac{1}{4}(\theta\Gamma^{ef})^\alpha(\theta\mathcal{R}_{ef}{}^{c_1\ldots c_4})_\beta\delta^{c_5}_{[a}\delta^{c_6}_{b]} + \frac{1}{2}(\theta\Gamma^{ef})^\alpha(\theta\mathcal{T}_{[a}{}^{c_1\ldots c_4}\mathcal{S}_{b]ef}{}^{c_5c_6})_\beta +\\ &\quad + 8(\theta\mathcal{T}_{[a}{}^{ec_1c_2c_3})^\alpha(\theta\mathcal{S}_{b]e}{}^{c_4c_5c_6})_\beta + 12(\theta\mathcal{T}_{[a}{}^{efc_5c_6})^\alpha(\theta\mathcal{T}_{b]}{}^{c_1\ldots c_4}\Gamma_{ef})_\beta -\\ &\quad - 12(\theta\mathcal{T}_{[a}{}^{efc_5c_6}\mathcal{T}_{b]}{}^{c_1\ldots c_4})^\alpha(\theta\Gamma_{ef})_\beta - 12(\theta\mathcal{T}_{[a}{}^{c_1\ldots c_4}\mathcal{T}_{b]}{}^{efc_5c_6})^\alpha(\theta\Gamma_{ef})_\beta\end{aligned} \tag{4.3}$$

This formula is most easily derived as follows: act with $\nabla_{[\alpha_1}\nabla_{\alpha_2]}$ on $T_{ab}{}^\alpha$, taking (2.8) into account; substitute the result in (3.21) for $S \to T_{ab}{}^\alpha$, $r \to 2$.

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4.2 Ω_M, $E_M{}^A$ and $E_A{}^M$

We now turn to the derivation of recursion relations for the components of the connection and the vielbein. The first line of definition (2.1) implies

$$2\partial_{(\mu}E_{\nu)}{}^a = T_{\mu\nu}{}^a - 2\Omega_{(\mu|e}{}^a E_{|\nu)}{}^e\,. \tag{4.4}$$

Multiplying both sides by θ^μ, taking (3.12), (3.13) into account, gives

$$E^{(n+1)a}_{\mu} = \frac{1}{n+2}\theta^\nu T_{\nu\mu}{}^a\,, \qquad n \geq 0\,. \tag{4.5}$$

Moreover, using (2.3), (2.18) we arrive at

$$\boxed{E^{(n+1)a}_{\mu} = -\frac{i}{n+2}E^{(n)\alpha}_{\mu}(\Gamma^a\theta)_\alpha\,, \qquad n \geq 0\,.} \tag{4.6}$$

Similarly, we can show that

$$\boxed{E^{(n+1)a}_{m} = -\frac{i}{n+1}E^{(n)\,\alpha}_{m}(\Gamma^a\theta)_\alpha\,, \qquad n \geq 0\,.} \tag{4.7}$$

The second line of definition (2.1) implies

$$2\partial_{(\mu}\Omega_{\nu)a}{}^b = R_{\mu\nu a}{}^b + 2\Omega_{(\mu|a}{}^c\Omega_{|\nu)c}{}^b\,. \tag{4.8}$$

Multiplying both sides by θ^μ gives

$$\begin{aligned}\Omega^{(n+1)}_{\mu ab} &= \frac{1}{n+2}\;\theta^\nu R^{(n)}{}_{\nu\mu ab}\\ &= \frac{i}{n+2}\sum_{r=0}^{n}\{E^{(r)\,\alpha}_{\mu}(\mathcal{R}_{ab}{}^{cdef}\theta)_\alpha G^{(n-r)}_{cdef} + E^{(r)\,e}_{\mu}(\theta\mathcal{S}_{eab}{}^{cd}T^{(n-r)}_{cd})\}, \qquad n \geq 0\,.\end{aligned} \tag{4.9}$$

The first equality is shown by taking (3.13) into account. The second one is a consequence of (2.29), (3.12), (2.5). Furthermore, using the recursion relations (4.1),(4.6) yields

$$\begin{aligned}\Omega^{(n+1)}_{\mu ab} &= \frac{i}{n+2}E^{(n)\alpha}_{\mu}(\theta\mathcal{R}_{ab}{}^{cdef})_\alpha G^{(0)}_{cdef} +\\ &+\frac{1}{n+2}\sum_{r=0}^{n-1}E^{(r)\,\alpha}_{\mu}\left(\frac{1}{n-r}C_{1ab}{}^{ef} + \frac{1}{r+2}C_{2ab}{}^{ef}\right)_{\alpha\beta}T^{(n-r-1)\beta}_{ef}\,, \qquad n \geq 0\,,\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}(C_{1ab}{}^{ef})_{\alpha\beta} &:= -6(\theta\mathcal{R}_{ab}{}^{cdef})_\alpha(\theta\Gamma_{cd})_\beta\\ (C_{2ab}{}^{ef})_{\alpha\beta} &:= (\theta\Gamma^g)_\alpha(\theta\mathcal{S}_{gab}{}^{ef})_\beta\,.\end{aligned} \tag{4.11}$$

Note that $\theta^\mu\Omega_\mu = 0$, as it should. This follows from (3.12) and the symmetry of $\mathcal{R}$.

Similarly one can show that

$$
\begin{aligned}
\Omega^{(n+1)}_{\ \ mab} &= \frac{i}{n+1}(\theta \mathcal{S}_{mab}{}^{ef} T^{(n)}_{\ \ ef}) + \frac{i}{n+1} E^{(n)\ \alpha}_{\ m} (\theta \mathcal{R}_{ab}{}^{cdef})_\alpha G^{(0)}_{cdef} + \\
&+ \frac{1}{n+1} \sum_{r=0}^{n-1} E^{(r)\ \alpha}_{\ m} \left(\frac{1}{n-r} C_{1ab}{}^{ef} + \frac{1}{r+1} C_{2ab}{}^{ef} \right)_{\alpha\beta} T^{(n-r-1)\beta}_{\ \ ef}, \qquad n \geq 0
\end{aligned} \tag{4.12}
$$

We are now ready to obtain recursion relations for the remaining components of the vielbein. The first line of definition (2.1) implies

$$
2\partial_{(\mu} E_{\nu)}{}^\alpha = T_{\mu\nu}{}^\alpha - 2\Omega_{(\mu|\beta}{}^\alpha E_{\nu)}{}^\beta . \tag{4.13}
$$

Multiplying both sides by θ^μ, taking (3.12), (3.13) into account, gives

$$
\begin{aligned}
E^{(n+1)\alpha}_{\ \ \mu} &= \frac{1}{n+2} \left(\theta^\nu T^{(n)\ \ \alpha}_{\ \nu\mu} - \theta^\beta \Omega^{(n)\ \ \alpha}_{\ \mu\beta} \right) \\
&= \frac{1}{n+2}\, \theta^\beta \left(\sum_{r=0}^{n} E^{(r)a}_{\ \nu} T^{(n-r)\alpha}_{\ \ a\beta} - \Omega^{(n)\ \ \alpha}_{\ \mu\beta} \right), \qquad n \geq 0 .
\end{aligned} \tag{4.14}
$$

The second equality follows from (2.18). For $n = 0$ in particular we have

$$
\boxed{E^{(1)\alpha}_{\ \ \mu} = 0 ,} \tag{4.15}
$$

as can be seen from (3.8), (3.10). For $n \geq 1$, we can further reduce (4.14) by using (4.10), (4.6), (2.4)

$$
\boxed{
\begin{aligned}
E^{(n+1)\alpha}_{\ \ \mu} &= \frac{i}{(n+1)(n+2)} E^{(n-1)\beta}_{\ \ \mu} (D_1^{cdef})_\beta{}^\alpha G^{(0)}_{cdef} + \\
&+ \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n-2} E^{(r)\beta}_{\ \mu} \left(\frac{1}{n-r-1} F_1^{ef} + \frac{1}{r+2} F_2^{ef} + \right. \\
&\qquad \left. + \frac{n+1}{(n-r-1)(r+2)} F_3^{ef} \right)^\alpha{}_{\beta\gamma} T^{(n-r-2)\ \gamma}_{\ \ ef}, \ n \geq 1 ,
\end{aligned}
}
\tag{4.16}
$$

where

$$
\begin{aligned}
(D_1^{cdef})_\beta{}^\alpha &:= \frac{1}{4} (\theta \mathcal{R}_{ab}{}^{cdef})_\beta (\theta \Gamma^{ab})^\alpha + (\theta \Gamma^a)_\beta (\theta \mathcal{T}_a{}^{cdef})^\alpha \\
(F_1^{ef})^\alpha{}_{\beta\gamma} &:= \frac{3}{2} (\theta \Gamma^{ab})^\alpha (\theta \mathcal{R}_{ab}{}^{cdef})_\beta (\theta \Gamma_{cd})_\gamma \\
(F_2^{ef})^\alpha{}_{\beta\gamma} &:= -\frac{1}{4} (\theta \Gamma^{ab})^\alpha (\theta \Gamma^g)_\beta (\theta \mathcal{S}_{gab}{}^{ef})_\gamma \\
(F_3^{ef})^\alpha{}_{\beta\gamma} &:= 6 (\theta \mathcal{T}_a{}^{bcef})^\alpha (\theta \Gamma^a)_\beta (\theta \Gamma_{bc})_\gamma .
\end{aligned} \tag{4.17}
$$

Note that for $n > 0$ it follows that $\theta^\mu E^{(n)\alpha}_{\ \ \mu} = 0$, as it should.

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We proceed similarly to arrive at the following recursion relations:

$$\boxed{E^{(1)}{}_m{}^\alpha = \frac{1}{4}(\theta\Gamma^{ab})^\alpha\omega_{mab} - (\theta\mathcal{T}_m{}^{cdef})^\alpha G^{(0)}_{cdef}} \tag{4.18}$$

and

$$\boxed{\begin{aligned} E^{(n+1)}{}_m{}^\alpha = &\frac{i}{n(n+1)}E^{(n-1)}{}_m{}^\beta\left(D_1^{cdef}\right)_\beta{}^\alpha G^{(0)}_{cdef} + \frac{i}{n(n+1)}T^{(n-1)}{}_{ef}{}^\beta\left(D_{2m}{}^{ef}\right)_\beta{}^\alpha + \\ &+\frac{1}{n(n+1)}\sum_{r=0}^{n-2}E^{(r)}{}_m{}^\beta\Bigg(\frac{1}{n-r-1}F_1^{ef} + \frac{1}{r+1}F_2^{ef} + \\ &\qquad\qquad + \frac{n}{(n-r-1)(r+1)}F_3^{ef}\Bigg)^\alpha{}_{\beta\gamma}T^{(n-r-2)}{}_{ef}{}^\gamma\,, \quad n\geq 1\,, \end{aligned}} \tag{4.19}$$

where

$$(D_{2a}{}^{bc})_\beta{}^\alpha := -\frac{1}{4}(\theta\mathcal{S}_{aef}{}^{bc})_\beta(\theta\Gamma^{ef})^\alpha + 6(\theta\Gamma_{ef})_\beta(\theta\mathcal{T}_a{}^{bcef})^\alpha\,. \tag{4.20}$$

Finally, note that if $E_M{}^A$ is known to order θ^n and $E_A{}^M$ is known to order θ^{n-1}, then equations (3.17) can be used to derive $E_A{}^M$ to order θ^n.

4.3 C_{MNP}

Let us turn to the recursion relations for the components of the three-form superpotential. The results of this section will be relevant to the derivation of the membrane action in section 7.

By definition, the C-field satisfies

$$4\partial_{[M}C_{NPQ\}} = G_{MNPQ}\,. \tag{4.21}$$

Up to a gauge choice, the following is a solution of (4.21) at each order in the θ expansion

$$\begin{aligned} C^{(0)}_{\mu\nu\sigma} &= C^{(0)}_{\mu\nu s} = C^{(0)}_{\sigma mn} = 0\,, \\ 4\partial_{[m}C^{(0)}_{npq]} &= G^{(0)}_{mnpq} \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} C^{(n+1)}_{\mu\nu\sigma} &= \frac{1}{n+4}\,\theta^\lambda G^{(n)}_{\lambda\mu\nu\sigma} \\ C^{(n+1)}_{\mu\nu s} &= \frac{1}{n+3}\,\theta^\lambda G^{(n)}_{\lambda\mu\nu s} \\ C^{(n+1)}_{\sigma mn} &= \frac{1}{n+2}\,\theta^\lambda G^{(n)}_{\lambda\sigma mn} \\ C^{(n+1)}_{mnp} &= \frac{1}{n+1}\,\theta^\lambda G^{(n)}_{\lambda mnp}\,, \qquad n\geq 0\,. \end{aligned} \tag{4.23}$$

Together with (2.12), equations (4.22), (4.23) are enough to determine the θ expansion of the C-field.

The right-hand sides of the formulae above can be further reduced by taking (2.12), (2.11), (3.12) into account. Explicitly we find

$$
\begin{aligned}
\theta^\lambda G_{\lambda\mu\nu\sigma} &= -3iE_{(\mu}{}^a E_\nu{}^b E_{\sigma)}{}^\delta (\Gamma_{ab}\theta)_\delta \\
\theta^\lambda G_{\lambda\mu\nu s} &= -iE_\mu{}^a E_\nu{}^b E_s{}^\gamma (\Gamma_{ab}\theta)_\gamma - 2iE_s{}^a E_{(\mu}{}^b E_{\nu)}{}^\gamma (\Gamma_{ab}\theta)_\gamma \\
\theta^\lambda G_{\lambda\sigma mn} &= -iE_m{}^a E_n{}^b E_\sigma{}^\delta (\Gamma_{ab}\theta)_\delta - 2iE_\sigma{}^a E_{[m}{}^b E_{n]}{}^\gamma (\Gamma_{ab}\theta)_\gamma \\
\theta^\lambda G_{\lambda mnp} &= -3iE_{[m}{}^a E_n{}^b E_{p]}{}^\gamma (\Gamma_{ab}\theta)_\gamma \,.
\end{aligned}
\tag{4.24}
$$

4.4 Maximally-supersymmetric superspaces

The recursion relations of the previous section can be solved iteratively, order-by-order in an expansion in powers of θ. We carry out this procedure in the next section, up to order $\mathcal{O}(\theta^5)$. As mentioned in the introduction, around flat space a 'dual' perturbative expansion is also possible: exact in θ but perturbative in powers of the background fields. This is explained in detail in section 6. However, it has been known for some time that in the special case of maximally-supersymmetric bosonic backgrounds of the type $AdS_d \times S^{D-d}$ one can obtain exact, closed expressions for the supervielbien components [27]–[31]. We now rederive this result using the methods of the present paper.

We first note that in a bosonic background $T^{(0)}_{ab}{}^\alpha$ vanishes. Moreover, using (4.1), (2.25) it can be seen that the first θ level of the gravitino field strength is given by

$$
\begin{aligned}
T^{(1)}_{ab}{}^\alpha &= e_a{}^m e_b{}^n \Big\{ \frac{1}{4}(\theta\Gamma^{pq})^\alpha R(\omega)_{mnpq} + 2(\theta \mathcal{T}_{[m}{}^{pqrs})^\alpha (\mathcal{D}_{n]} G_{pqrs}) \\
&\qquad\qquad - 2(\theta \mathcal{T}_{[m}{}^{pqrs} \mathcal{T}_{n]}{}^{p'q'r's'})^\alpha G_{pqrs} G_{p'q'r's'} \Big\} \\
&= e_a{}^m e_b{}^n \theta^\beta (\mathcal{R}^{Tr}_{mn})_\beta{}^\alpha \,,
\end{aligned}
\tag{4.25}
$$

where $G_{pqrs} := 4\partial_{[p} C^{(0)}_{qrs]}$ and $(\mathcal{R}_{mn})^\alpha{}_\beta := ([\mathbb{D}_m, \mathbb{D}_n])^\alpha{}_\beta$ is the curvature of the supercovariant derivative

$$
(\mathbb{D}_m)^\alpha{}_\beta := (\mathcal{D}_m)^\alpha{}_\beta - (\mathcal{T}^{Tr}_m{}^{pqrs})^\alpha{}_\beta G_{pqrs} \,. \tag{4.26}
$$

Note that in a bosonic background $\omega = \overset{0}{\omega}$ and $\mathcal{D} = \overset{0}{\mathcal{D}}$. Killing spinors are parallel with respect to $\mathbb{D}$. It follows that Killing spinors are eigenverctors of the supercurvature $(\mathcal{R}_{mn})$ with zero eigenvalue and therefore in a maximally-supersymmetric space

$$
(\mathcal{R}_{mn})^\alpha{}_\beta = 0 \,, \tag{4.27}
$$

so that $T^{(1)}_{ab}{}^\alpha$ vanishes by (4.25). All possible solutions to (4.27) were classified in [41] up to local isometries. These are: $\mathbb{R}^{1,10}$, $AdS_4 \times S^7$, $AdS_7 \times S^4$, and the Hpp-wave.

Since $T^{(n)}_{ab}{}^\alpha$ is zero for $n = 0, 1$, it now follows by induction from (4.2) that $T_{ab}{}^\alpha$ vanishes identically. Hence in the special case of maximally-supersymmetric spaces the gravitino field-strength drops out of the recursion relations (4.16), (4.19), which can now be solved straightforwardly. The result is

$$
\begin{aligned}
E_\mu{}^\alpha &= \delta_\mu{}^\beta [\mathcal{P}^{-1/2} \sinh \sqrt{\mathcal{P}}]_\beta{}^\alpha \\
E_m{}^\alpha &= E^{(1)}_m{}^\beta [\mathcal{P}^{-1/2} \sinh \sqrt{\mathcal{P}}]_\beta{}^\alpha \,,
\end{aligned}
\tag{4.28}
$$

where

$$[\mathcal{P}]_\alpha{}^\beta := i(D_1^{mnpq})_\alpha{}^\beta G_{mnpq}\,, \tag{4.29}$$

and the functions of $\mathcal{P}$ above are defined formally by their Taylor expansions around zero. The expressions for $D_1^{mnpq} := e_a{}^m e_b{}^n e_c{}^p e_d{}^q D_1^{abcd}$ and $E^{(1)}{}_m{}^\alpha$ were given in (4.17), (4.18) respectively. The remaining components of the supervielbein follow similarly from (4.6), (4.7):

$$\begin{aligned} E_\mu{}^a &= 2i\delta_\mu{}^\beta \left[\mathcal{P}^{-1}\sinh^2\frac{\sqrt{\mathcal{P}}}{2}\right]_\beta{}^\alpha(\Gamma^a\theta)_\alpha \\ E_m{}^a &= e_m{}^a + 2iE^{(1)}{}_m{}^\beta\left[\mathcal{P}^{-1}\sinh^2\frac{\sqrt{\mathcal{P}}}{2}\right]_\beta{}^\alpha(\Gamma^a\theta)_\alpha\,. \end{aligned} \tag{4.30}$$

It can be seen that equations (4.28), (4.30) above are the same as (up to conventions) of [27, eqs. (22), (23)], or [29, eq. (3.9)][3]. The method presented here like [30] does not rely on coset-superspace techniques. As is clear from the derivation, formulae (4.28), (4.30) are valid for all eleven-dimensional maximally-supersymmetric bosonic backgrounds.

5. Expansion up to $\mathcal{O}(\theta^5)$

We now give the explicit expressions for the first few terms in the θ-expansion of the vielbein and the C-field. Explicit (partial) results to order θ^2 have appeared in [32] (see also [42]). Our results are in complete agreement with those of [32] (cf. equations (4.15), (4.16) of that reference) to the order they have computed.[4]

The authors of [33] have computed the vielbein expansion to $\mathcal{O}(\theta^3)$, but have omitted some tedious (albeit straightforward) steps that would facilitate the comparison of their expressions to the explicit formulae in this section. After some algebra, formulae [33, (5.7)–(5.10)] can be seen to agree with our results (to the order they have computed), apart from signs that may be attributed to different conventions.[5]

In the following we will need the explicit expressions for $T^{(1)}$, $T^{(2)}$. These can be read off from (4.1), (4.2):

$$\begin{aligned} T^{(1)}{}_{ab}{}^\alpha = {}& \frac{1}{4}(\theta\Gamma^{cd})^\alpha R^{(0)}_{abcd} + 2(\theta\mathcal{T}_{[a}{}^{cdef})^\alpha(\nabla_{b]}G_{cdef})^{(0)} - \\ & - 2(\theta\mathcal{T}_{[a}{}^{cdef}\mathcal{T}_{b]}{}^{c'd'e'f'})^\alpha G^{(0)}_{cdef}G^{(0)}_{c'd'e'f'}\,, \end{aligned} \tag{5.1}$$

---

[3] The matrix $\mathcal{M}^2$ in those references is proportional to the matrix $\mathcal{P}$ of the present paper.

[4] In order to translate the expressions of [32] to our conventions, one needs to make the following substitutions: $\theta \to \frac{1}{\sqrt{2}}\theta$; $\bar\theta \to \frac{i}{\sqrt{2}}\theta^{Tr}C$; $\Psi_m \to \frac{1}{\sqrt{2}}\Psi_m$ and $E_\mu{}^a \to \sqrt{2}E_\mu{}^a$; $E_m{}^\alpha \to \frac{1}{\sqrt{2}}E_m{}^\alpha$; $B_{mnp} \to -C_{mnp}$; $B_{mn\sigma} \to -\sqrt{2}C_{mn\sigma}$; $B_{m\nu\sigma} \to -2C_{m\nu\sigma}$; $B_{\mu\nu\sigma} \to -2\sqrt{2}C_{\mu\nu\sigma}$; $\hat{F}_{mnpq} \to -G^{(0)}_{mnpq}$; $T_s{}^{mnpq} \to (\mathcal{T}^{Tr})_s{}^{mnpq}$.

[5] Our θ is equivalent to y of [33]. Note that there seem to be the following typos in [33]: $\mathcal{O}(y^5)$ in (5.8) should be replaced by $\mathcal{O}(y^4)$, $\mathcal{O}(y^4)$ in (5.10) should be replaced by $\mathcal{O}(y^3)$, the factor of 1/2 in front of the second term on the right-hand side of (5.10) should be replaced by 1/6, the i in front of the second term on the right-hand side of (5.9) should be deleted.

where

$$
\begin{aligned}
(\nabla_b G_{cdef})^{(0)} &= e_b{}^m \mathcal{D}_m G^{(0)}_{cdef} + E^{(0)}{}_b^{\mu} \partial_\mu G^{(1)}_{cdef} \\
&= e_b{}^m \mathcal{D}_m G^{(0)}_{cdef} - 6i(\Psi_b \Gamma_{[cd} T^{(0)}_{ef]})
\end{aligned}
\tag{5.2}
$$

and

$$
T^{(2)}_{ab}{}^{\alpha} = \frac{i}{2}(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}(\nabla_{|b]} T_{ef}{}^{\beta})^{(0)} + \frac{i}{2}(\mathcal{N}_{ab}{}^{c_1\ldots c_6})^{\alpha}{}_{\beta} G^{(0)}_{c_1\ldots c_4} T^{(0)}_{c_5c_6}{}^{\beta}\,,
\tag{5.3}
$$

where

$$
\begin{aligned}
(\nabla_b T_{ef}{}^{\beta})^{(0)} &= e_b{}^m \mathcal{D}_m T^{(0)}_{ef}{}^{\beta} + E^{(0)}{}_b^{\mu} \partial_\mu T^{(1)}_{ef}{}^{\beta} \\
&= e_b{}^m \mathcal{D}_m T^{(0)}_{ef}{}^{\beta} - \frac{1}{4}(\Psi_b \Gamma^{cd})^{\beta} R^{(0)}_{efcd} - \\
&\quad - 2(\Psi_b \mathcal{T}_{[e}{}^{cdgh})^{\beta} e_{f]}{}^m \mathcal{D}_m G^{(0)}_{cdgh} + 12i(\Psi_b \mathcal{T}_{[e}{}^{cdgh})^{\beta}(\Psi_{f]}\Gamma_{cd} T^{(0)}_{gh}) + \\
&\quad + 2(\Psi_b \mathcal{T}_{[e}{}^{cdgh} T_{f]}{}^{c'd'g'h'})^{\beta} G^{(0)}_{cdgh} G^{(0)}_{c'd'g'h'}\,.
\end{aligned}
\tag{5.4}
$$

Vielbein expansions

Using formulae (4.6), (4.7), (4.15), (4.16), (4.18), (4.19) we find for the first few terms

$$
\begin{aligned}
E^{(0)}{}_{\mu}{}^{a} &= 0 \\
E^{(1)}{}_{\mu}{}^{a} &= -\frac{i}{2}(\Gamma^a\theta)_\mu \\
E^{(2)}{}_{\mu}{}^{a} &= 0 \\
E^{(3)}{}_{\mu}{}^{a} &= \frac{1}{24}(D_1^{cdef}\Gamma^a\theta)_\mu G^{(0)}_{cdef} \\
E^{(4)}{}_{\mu}{}^{a} &= \frac{i}{120}\Big(2F_1^{ef} + F_2^{ef} + 3F_3^{ef}\Big)^{\alpha}{}_{\mu\beta}(\Gamma^a\theta)_\alpha T^{(0)\beta}_{ef} \\
E^{(5)}{}_{\mu}{}^{a} &= \frac{i}{720}(D_1^{cdef}D_1^{c'd'e'f'}\Gamma^a\theta)_\mu G^{(0)}_{cdef}G^{(0)}_{c'd'e'f'} + \frac{i}{240}\Big(F_1^{ef}+F_2^{ef}+2F_3^{ef}\Big)^{\alpha}{}_{\mu\beta}(\Gamma^a\theta)_\alpha T^{(1)\beta}_{ef}\,,
\end{aligned}
\tag{5.5}
$$

$$
\begin{aligned}
E^{(0)}{}_{m}{}^{a} &= e_m{}^a \\
E^{(1)}{}_{m}{}^{a} &= -i(\Psi_m\Gamma^a\theta) \\
E^{(2)}{}_{m}{}^{a} &= -\frac{i}{8}(\theta\Gamma^{aef}\theta)\omega_{mef} + \frac{i}{2}(\theta\mathcal{T}_m{}^{cdef}\Gamma^a\theta)G^{(0)}_{cdef} \\
E^{(3)}{}_{m}{}^{a} &= \frac{1}{6}(\Psi_m D_1^{cdef}\Gamma^a\theta)G^{(0)}_{cdef} + \frac{1}{6}(T^{(0)}_{ef}D_{2m}{}^{ef}\Gamma^a\theta) \\
E^{(4)}{}_{m}{}^{a} &= \frac{1}{96}(\theta\Gamma^{gh}D_1^{cdef}\Gamma^a\theta)\omega_{mgh}G^{(0)}_{cdef} - \frac{1}{24}(\theta\mathcal{T}_m{}^{cdef}D_1^{c'd'e'f'}\Gamma^a\theta)G^{(0)}_{cdef}G^{(0)}_{c'd'e'f'} + \\
&\quad + \frac{1}{24}(T^{(1)}_{ef}D_{2m}{}^{ef}\Gamma^a\theta) + \frac{i}{24}\Big(F_1^{ef} + F_2^{ef} + 2F_3^{ef}\Big)^{\alpha}{}_{\beta\gamma}(\Gamma^a\theta)_\alpha\Psi_m{}^{\beta}T^{(0)\gamma}_{ef} \\
E^{(5)}{}_{m}{}^{a} &= \frac{i}{120}(\Psi_m D_1^{cdef}D_1^{c'd'e'f'}\Gamma^a\theta)G^{(0)}_{cdef}G^{(0)}_{c'd'e'f'} + \frac{i}{120}(T^{(0)}_{ef}D_{2m}{}^{ef}D_1^{cdgh}\Gamma^a\theta)G^{(0)}_{cdgh} + \\
&\quad + \frac{i}{120}\Big(F_1^{ef} + 2F_2^{ef} + 3F_3^{ef}\Big)^{\alpha}{}_{\beta\gamma}(\Gamma^a\theta)_\alpha\Psi_m{}^{\beta}T^{(1)\gamma}_{ef} + \frac{1}{60}(T^{(2)}_{ef}D_{2m}{}^{ef}\Gamma^a\theta) + \\
&\quad + \frac{i}{120}\Big(2F_1^{ef} + F_2^{ef} + 3F_3^{ef}\Big)^{\alpha}{}_{\beta\gamma}(\Gamma^a\theta)_\alpha(\frac{1}{4}\theta\Gamma^{gh}\omega_{mgh} - \theta\mathcal{T}_m{}^{cdgh}G^{(0)}_{cdgh})^{\beta}T^{(0)\gamma}_{ef}
\end{aligned}
\tag{5.6}
$$

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and

$$
\begin{aligned}
E^{(0)}{}_\mu{}^\alpha &= \delta_\mu{}^\alpha \\
E^{(1)}{}_\mu{}^\alpha &= 0 \\
E^{(2)}{}_\mu{}^\alpha &= \frac{i}{6}(D_1^{cdef})_\mu{}^\alpha G^{(0)}_{cdef} \\
E^{(3)}{}_\mu{}^\alpha &= \frac{1}{24}\Big(2F_1^{ef} + F_2^{ef} + 3F_3^{ef}\Big)^\alpha{}_{\mu\beta} T^{(0)\beta}_{ef} \\
E^{(4)}{}_\mu{}^\alpha &= -\frac{1}{120}(D_1^{cdef} D_1^{c'd'e'f'})_\mu{}^\alpha G^{(0)}_{cdef} G^{(0)}_{c'd'e'f'} + \frac{1}{40}\Big(F_1^{ef} + F_2^{ef} + 2F_3^{ef}\Big)^\alpha{}_{\mu\beta} T^{(1)\beta}_{ef}\,,
\end{aligned} \tag{5.7}
$$

$$
\begin{aligned}
E^{(0)}{}_m{}^\alpha &= \Psi_m{}^\alpha \\
E^{(1)}{}_m{}^\alpha &= \frac{1}{4}(\theta\Gamma^{ef})^\alpha \omega_{mef} - (\theta\mathcal{T}_m{}^{cdef})^\alpha G^{(0)}_{cdef} \\
E^{(2)}{}_m{}^\alpha &= \frac{i}{2}(\Psi_m D_1^{cdef})^\alpha G^{(0)}_{cdef} + \frac{i}{2}(T^{(0)}_{ef} D_{2m}{}^{ef})^\alpha \\
E^{(3)}{}_m{}^\alpha &= \frac{i}{24}(\theta\Gamma^{ab} D_1^{cdef})^\alpha \omega_{mab} G^{(0)}_{cdef} - \frac{i}{6}(\theta\mathcal{T}_m{}^{cdef} D_1^{c'd'e'f'})^\alpha G^{(0)}_{cdef} G^{(0)}_{c'd'e'f'} + \\
&\quad + \frac{i}{6}(T^{(1)}_{ef} D_{2m}{}^{ef})^\alpha - \frac{1}{6}\Big(F_1^{ef} + F_2^{ef} + 2F_3^{ef}\Big)^\alpha{}_{\beta\gamma} \Psi_m{}^\beta T^{(0)\gamma}_{ef} \\
E^{(4)}{}_m{}^\alpha &= -\frac{1}{24}(\Psi_m D_1^{cdef} D_1^{c'd'e'f'})^\alpha G^{(0)}_{cdef} G^{(0)}_{c'd'e'f'} - \frac{1}{24}(T^{(0)}_{ef} D_{2m}{}^{ef} D_1^{cdgh})^\alpha G^{(0)}_{cdgh} - \\
&\quad - \frac{1}{24}\Big(F_1^{ef} + 2F_2^{ef} + 3F_3^{ef}\Big)^\alpha{}_{\beta\gamma} \Psi_m{}^\beta T^{(1)\gamma}_{ef} + \frac{i}{12}(T^{(2)}_{ef} D_{2m}{}^{ef})^\alpha - \\
&\quad - \frac{1}{24}\Big(2F_1^{ef} + F_2^{ef} + 3F_3^{ef}\Big)^\alpha{}_{\beta\gamma}\big(\frac{1}{4}\theta\Gamma^{ab}\omega_{mab} - \theta\mathcal{T}_m{}^{cdgh} G^{(0)}_{cdgh}\big)^\beta T^{(0)\gamma}_{ef}\,.
\end{aligned} \tag{5.8}
$$

C-field expansions

$$
\begin{aligned}
C^{(0)}_{\mu\nu\sigma} &= 0 \\
C^{(1)}_{\mu\nu\sigma} &= 0 \\
C^{(2)}_{\mu\nu\sigma} &= 0 \\
C^{(3)}_{\mu\nu\sigma} &= \frac{i}{8}(\theta\Gamma_{ab})_{(\mu}(\theta\Gamma^a)_\nu(\theta\Gamma^b)_{\sigma)} \\
C^{(4)}_{\mu\nu\sigma} &= 0\,,
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
C^{(0)}_{\mu\nu s} &= 0 \\
C^{(1)}_{\mu\nu s} &= 0 \\
C^{(2)}_{\mu\nu s} &= \frac{1}{4}(\theta\Gamma_{se})_{(\mu}(\theta\Gamma^e)_{\nu)} \\
C^{(3)}_{\mu\nu s} &= \frac{i}{20}(\theta\Gamma^a)_\mu(\theta\Gamma^b)_\nu(\Psi_s\Gamma_{ab}\theta) + \frac{i}{5}(\theta\Gamma_{ab})_{(\mu}(\theta\Gamma^a)_{\nu)}(\Psi_s\Gamma^b\theta) \\
C^{(4)}_{\mu\nu s} &= \frac{i}{24}(\theta\Gamma^a)_\mu(\theta\Gamma^b)_\nu\left\{-\frac{1}{2}(\theta\theta)\omega_{sab} + \frac{1}{4}(\theta\Gamma_{abcd}\theta)\omega_s{}^{cd} - (\theta\mathcal{T}_s{}^{cdef}\Gamma_{ab}\theta)G^{(0)}_{cdef}\right\} + \\
&\quad + \frac{i}{12}(\theta\Gamma_{ab})_{(\mu}(\theta\Gamma^a)_{\nu)}\left\{\frac{1}{4}(\theta\Gamma^{bef}\theta)\omega_{sef} - (\theta\mathcal{T}_s{}^{cdef}\Gamma^b\theta)G^{(0)}_{cdef}\right\} - \\
&\quad - \frac{i}{72}(D_1^{abcd}\Gamma^e\theta)_{(\mu|}(\theta\Gamma_{se})_{|\nu)} G^{(0)}_{abcd} + \frac{i}{36}(D_1^{abcd}\Gamma_{se}\theta)_{(\mu}(\theta\Gamma^e)_{\nu)} G^{(0)}_{abcd}\,,
\end{aligned} \tag{5.10}
$$

$$
\begin{aligned}
C^{(0)}_{\sigma mn} &= 0 \\
C^{(1)}_{\sigma mn} &= -\frac{i}{2}(\theta\Gamma_{mn})_\sigma \\
C^{(2)}_{\sigma mn} &= \frac{2}{3}(\Psi_{[m}\Gamma^e\theta)(\theta\Gamma_{n]e})_\sigma - \frac{1}{3}(\Psi_{[m}\Gamma_{n]e}\theta)(\theta\Gamma^e)_\sigma \\
C^{(3)}_{\sigma mn} &= \frac{1}{4}\left\{\frac{1}{4}(\theta\Gamma^{egh}\theta)\omega_{[m|gh} - (\theta\mathcal{T}_{[m|}{}^{cdgh}\Gamma^e\theta)G^{(0)}_{cdgh}\right\}(\theta\Gamma_{|n]e})_\sigma - \\
&\quad - \frac{1}{4}\left\{\frac{1}{4}(\theta\Gamma^{gh}{}_{e[m}\theta)\omega_{n]gh} - \frac{1}{2}(\theta\theta)\omega_{[mn]e} - (\theta\mathcal{T}_{[m}{}^{cdgh}\Gamma_{n]e}\theta)G^{(0)}_{cdgh}\right\}(\theta\Gamma^e)_\sigma + \\
&\quad + \frac{i}{4}(\Psi_m\Gamma^a\theta)(\Psi_n\Gamma^b\theta)(\theta\Gamma_{ab})_\sigma - \frac{i}{4}(\Psi_{[m}\Gamma^a\theta)(\Psi_{n]}\Gamma_{ab}\theta)(\theta\Gamma^b)_\sigma + \\
&\quad + \frac{1}{24}(D_1^{cdgh}\Gamma_{mn}\theta)_\sigma G^{(0)}_{cdgh} \\
C^{(4)}_{\sigma mn} &= \frac{i}{15}\left\{(\Psi_{[m|}D_1^{cdgh}\Gamma^e\theta)G^{(0)}_{cdgh} + (T^{(0)}_{gh}D_{2[m|}{}^{gh}\Gamma^e\theta)\right\}(\theta\Gamma_{|n]e})_\sigma + \\
&\quad + \frac{i}{5}\left\{\frac{1}{4}(\theta\Gamma^{agh}\theta)\omega_{[m|gh} - (\theta\mathcal{T}_{[m|}{}^{cdgh}\Gamma^a\theta)G^{(0)}_{cdgh}\right\}(\Psi_{|n]}\Gamma^b\theta)(\theta\Gamma_{ab})_\sigma + \\
&\quad + \frac{i}{15}(\Psi_{[m}\Gamma^e\theta)(D_1^{cdgh}\Gamma_{n]e}\theta)_\sigma G^{(0)}_{cdgh} - \frac{i}{60}(\Psi_{[m}\Gamma_{n]e}\theta)(D_1^{cdgh}\Gamma^e\theta)_\sigma G^{(0)}_{cdgh} - \\
&\quad - \frac{i}{10}\{(\Psi_{[m}D_1^{cdgh}\Gamma_{n]e}\theta)G^{(0)}_{cdgh} + (T^{(0)}_{gh}D_{2[m}{}^{gh}\Gamma_{n]e}\theta)\}(\theta\Gamma^e)_\sigma + \\
&\quad + \frac{i}{5}\left\{-\frac{1}{2}(\theta\theta)\omega_{[m|ab} + \frac{1}{4}(\theta\Gamma^{gh}{}_{ab}\theta)\omega_{[m|gh} - (\theta\mathcal{T}_{[m|}{}^{cdgh}\Gamma_{ab}\theta)G^{(0)}_{cdgh}\right\}\times \\
&\quad \times(\Psi_{|n]}\Gamma^a\theta)(\theta\Gamma^b)_\sigma - \\
&\quad - \frac{i}{10}\left\{\frac{1}{4}(\theta\Gamma^{agh}\theta)\omega_{[m|gh} - (\theta\mathcal{T}_{[m|}{}^{cdgh}\Gamma^a\theta)G^{(0)}_{cdgh}\right\}(\Psi_{|n]}\Gamma_{ab}\theta)(\theta\Gamma^b)_\sigma\,,
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
4\partial_{[s}C^{(0)}_{mnp]} &= G^{(0)}_{smnp} \\
C^{(1)}_{mnp} &= -3i(\Psi_{[m}\Gamma_{np]}\theta) \\
C^{(2)}_{mnp} &= -\frac{3i}{2}\left\{-\frac{1}{2}(\theta\theta)\omega_{[mnp]} + \frac{1}{4}(\theta\Gamma^{gh}{}_{[mn}\theta)\omega_{p]gh} - (\theta\mathcal{T}_{[m}{}^{cdgh}\Gamma_{np]}\theta)G^{(0)}_{cdgh}\right\} - \\
&\quad - 3(\Psi_{[m}\Gamma^e\theta)(\Psi_n\Gamma_{p]e}\theta) \\
C^{(3)}_{mnp} &= \frac{1}{2}(\Psi_{[m}D_1^{cdgh}\Gamma_{np]}\theta)G^{(0)}_{cdgh} + \frac{1}{2}(T^{(0)}_{gh}D_{2[m}{}^{gh}\Gamma_{np]}\theta) + \\
&\quad + 2(\Psi_{[m|}\Gamma^e\theta)\times \\
&\quad \times\left\{\frac{1}{2}(\theta\theta)\omega_{|np]e} - \frac{1}{4}(\theta\Gamma^{gh}{}_{e|n}\theta)\omega_{p]gh} + (\theta\mathcal{T}_{|n}{}^{cdgh}\Gamma_{p]e}\theta)G^{(0)}_{cdgh}\right\} - \\
&\quad - (\Psi_{[m}\Gamma_{n|}{}^e\theta)\{\frac{1}{4}(\theta\Gamma^{gh}{}_e\theta)\omega_{|p]gh} - (\theta\mathcal{T}_{|p]}{}^{cdgh}\Gamma_e\theta)G^{(0)}_{cdgh}\} + \\
&\quad + i(\Psi_{[m}\Gamma^a\theta)(\Psi_n\Gamma^b\theta)(\Psi_{p]}\Gamma_{ab}\theta) \\
C^{(4)}_{mnp} &= \frac{1}{32}(\theta\Gamma^{gh}D_1^{cdef}\Gamma_{[mn}\theta)\omega_{p]gh}G^{(0)}_{cdef} -
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{8}(\theta\mathcal{T}_{[m}{}^{cdgh}D_1^{c'd'g'h'}\Gamma_{np]}\theta)G^{(0)}_{cdgh}G^{(0)}_{c'd'g'h'}+\\
&+\frac{1}{8}(T^{(1)}_{gh}D_{2[m}{}^{gh}\Gamma_{np]}\theta)+\frac{i}{8}(F_1^{gh}+F_2^{gh}+2F_3^{gh})^{\alpha}{}_{\beta\gamma}(\Gamma_{[mn}\theta)_{\alpha}\Psi_{p]}{}^{\beta}T^{(0)\gamma}_{gh}-\\
&-\frac{3i}{4}(\Psi_{[m|}\Gamma^{e}\theta)\{(\Psi_{|n}D_1^{cdgh}\Gamma_{p]e}\theta)G^{(0)}_{cdgh}+(T^{(0)}_{gh}D_{2|n}{}^{gh}\Gamma_{p]e}\theta)\}+\\
&+\frac{3i}{4}(\Psi_{[m}\Gamma^{a}\theta)(\Psi_{n|}\Gamma^{b}\theta)\times\\
&\times\left\{-\frac{1}{2}(\theta\theta)\omega_{|p]ab}+\frac{1}{4}(\theta\Gamma^{gh}{}_{ab}\theta)\omega_{|p]gh}-(\theta\mathcal{T}_{|p]}{}^{cdgh}\Gamma_{ab}\theta)G^{(0)}_{cdgh}\right\}-\\
&-\frac{i}{4}(\Psi_{[m}\Gamma_{n|e}\theta)\left\{(\Psi_{|p]}D_1^{cdgh}\Gamma^{e}\theta)G^{(0)}_{cdgh}+(T^{(0)}_{gh}D_{2|p]}{}^{gh}\Gamma^{e}\theta)\right\}-\\
&-\frac{3i}{4}(\Psi_{[m}\Gamma^{a}\theta)(\Psi_{n|}\Gamma_{ab}\theta)\left\{\frac{1}{4}(\theta\Gamma^{bgh}\theta)\omega_{|p]gh}-(\theta\mathcal{T}_{|p]}{}^{cdgh}\Gamma^{b}\theta)G^{(0)}_{cdgh}\right\}-\\
&-\frac{3}{4}\left\{\frac{1}{4}(\theta\Gamma^{egh}\theta)\omega_{[m|gh}-(\theta\mathcal{T}_{[m|}{}^{cdgh}\Gamma^{e}\theta)G^{(0)}_{cdgh}\right\}\times\\
&\times\left\{-\frac{1}{2}(\theta\theta)\omega_{|np]e}+\frac{1}{4}(\theta\Gamma^{g'h'}{}_{e|n}\theta)\omega_{p]g'h'}-(\theta\mathcal{T}_{|n}{}^{c'd'g'h'}\Gamma_{p]e}\theta)G^{(0)}_{c'd'g'h'}\right\}. \qquad (5.12)
\end{aligned}
$$

6. Expansion in the number of fields

In this section we expand our superspace geometry around the flat-space solution

$$
\begin{aligned}
h_m{}^{a} &:= e_m{}^{a}-\delta^{a}_{m}=0\\
\Psi_m{}^{\alpha} &= 0\\
C^{(0)}_{mnp} &= 0
\end{aligned} \qquad (6.1)
$$

and we show how to obtain results exact to all orders in the θ-expansion, but perturbative in the number of fields $h_m{}^a, \Psi_m{}^\alpha, C^{(0)}_{mnp}$. Note that in an expansion around flat space, the connection ω_m and the covariant field strengths $G^{(0)}_{abcd}$, $T^{(0)}_{ab}{}^{\alpha}$ and $R^{(0)}_{abcd}$ start at linear order in the fields (6.1). More precisely, from equations (2.20), (2.13), (2.27), (2.30) we see that

$$
\begin{aligned}
\omega_{nkm} &= \partial_{[k}h_{m]}{}^{a}\eta_{an}-\partial_{[m}h_{n]}{}^{a}\eta_{ak}-\partial_{[n}h_{k]}{}^{a}\eta_{am}+\cdots\\
G^{(0)}_{abcd} &= 4\delta_a{}^{m}\delta_b{}^{n}\delta_c{}^{p}\delta_d{}^{q}\partial_{[m}C^{(0)}_{npq]}\cdots\\
T^{(0)}_{ab}{}^{\alpha} &= 2\delta_a{}^{m}\delta_b{}^{n}\partial_{[m}\Psi_{n]}{}^{\alpha}+\cdots\\
R^{(0)}_{mnpq} &= \delta_a{}^{m}\delta_b{}^{n}\delta_c{}^{p}\delta_d{}^{q}R(\omega)_{mnpq}+\cdots\ ,
\end{aligned} \qquad (6.2)
$$

where the ellipses indicate terms quadratic or higher in the fields.

Going back to section 4 we split (4.2) as follows

$$
T^{(n)}_{ab}{}^{\alpha} = \frac{i}{n(n-1)}(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}\ \delta_{|b]}{}^{m}\partial_m T^{(n-2)}_{ef}{}^{\beta}+Q^{(n)}_{ab}{}^{\alpha}\,, \qquad (6.3)
$$

where the nonlinear piece Q reads

$$
\begin{aligned}
Q^{(n)}{}_{ab}{}^{\alpha} := & \frac{i}{n(n-1)}(\mathcal{N}_{ab}{}^{c_1\ldots c_6})^{\alpha}{}_{\beta}(G_{c_1\ldots c_4}T_{c_5c_6}{}^{\beta})^{(n-2)} - \\
& -\frac{i}{n(n-1)}(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}\; h_{|b]}{}^{m}\partial_m T^{(n-2)}{}_{ef}{}^{\beta} + \\
& +\frac{i}{n(n-1)}(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}\; e_{|b]}{}^{m}\sum_{r=0}^{n-2}(\Omega_m^{(r)}T^{(n-r-2)})_{ef}{}^{\beta} + \\
& +\frac{i}{n(n-1)}(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}\sum_{r=1}^{n-2}E^{(r)}{}_{|b]}{}^{m}(\nabla_m T_{ef}{}^{\beta})^{(n-r-2)} + \\
& +\frac{i}{n(n-1)}(\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta}\sum_{r=0}^{n-2}E^{(r)}{}_{|b]}{}^{\mu}(\nabla_\mu T_{ef}{}^{\beta})^{(n-r-2)} .
\end{aligned}
\tag{6.4}
$$

We have defined

$$
h_a{}^m := \delta_a^n h_n{}^b \delta_b^m \,, \tag{6.5}
$$

so that

$$
e_a{}^m = \delta_a^m - h_a{}^m + \mathcal{O}(h^2) \,. \tag{6.6}
$$

We can rewrite (6.3) more compactly using matrix notation and omitting spinor indices

$$
T^{(n)}_{ab} = \frac{1}{n(n-1)}[\mathcal{O}]_{ab}{}^{ef}T^{(n-2)}_{ef} + Q^{(n)}_{ab} \,, \tag{6.7}
$$

where

$$
[\mathcal{O}]_{ab}{}^{ef} := i[\mathcal{M}_{[a}{}^{ef}]\delta_{b]}{}^{m}\partial_m \,. \tag{6.8}
$$

The point of the split in (6.7) is that once $T^{(n)}$ is evaluated to any given order in the number of fields, $Q^{(n)}$ can be evaluated to the next order. This follows the fact that $Q^{(n)}$ starts at quadratic order in the number fields, as can be seen from definition (6.4).

Iterating (6.7) we obtain

$$
\begin{aligned}
T^{(2k)}_{ab} &= \frac{1}{(2k)!}[\mathcal{O}^k]_{ab}{}^{ef}T^{(0)}_{ef} + \sum_{r=0}^{k-1}\frac{(2k-2r)!}{(2k)!}[\mathcal{O}^r]_{ab}{}^{ef}Q^{(2k-2r)}_{ef} \\
T^{(2k+1)}_{ab} &= \frac{1}{(2k+1)!}[\mathcal{O}^k]_{ab}{}^{ef}T^{(1)}_{ef} + \sum_{r=0}^{k-1}\frac{(2k-2r+1)!}{(2k+1)!}[\mathcal{O}^r]_{ab}{}^{ef}Q^{(2k-2r+1)}_{ef} \,,
\end{aligned}
\tag{6.9}
$$

where $T^{(1)}$ was given explicitly in (5.1). Using the above results, and the equations of the previous section, the vielbein and the connection can be computed iteratively in the number of fields. The same is true for the inverse vielbein $E_A{}^M$, which can be seen as follows. Let us assume that $E_M{}^A$ is known to k-th order in the fields and that $E_A{}^M$ is known to $(k-1)$-th order. Equation (3.17) implies

$$
\begin{aligned}
E^{(n)}{}_{b}{}^{\mu} = & -e_b{}^m E^{(n)}{}_{m}{}^{\alpha}\delta_\alpha{}^\mu - \sum_{r=1}^{n-1}E^{(r)}{}_{b}{}^{m}E^{(n-r)}{}_{m}{}^{\alpha}\delta_\alpha{}^\mu - \\
& -\sum_{r=0}^{n-1}E^{(r)}{}_{b}{}^{\nu}E^{(n-r)}{}_{\nu}{}^{\alpha}\delta_\alpha{}^\mu + \sum_{r=0}^{n-1}E^{(r)}{}_{b}{}^{M}E^{(n-r)}{}_{M}{}^{a}\Psi_a{}^\mu \,.
\end{aligned}
\tag{6.10}
$$

Clearly, the first term on the right-hand-side is known to k-th order. The remaining terms on the right-hand-side are also known to k-th order, because they start at quadratic order in the fields. Hence $E^{(n)}{}_b{}^\mu$ can be computed to k-th order, using the formula above.

Similarly, for $E_b{}^m$ we have

$$\begin{aligned} E^{(n)}{}_b{}^m &= -e_b{}^p E^{(n)}{}_p{}^a e_a{}^m + \frac{i}{2} E^{(n-1)}{}_b{}^\mu (\Gamma^m\theta)_\mu - \\ &\quad - \sum_{r=1}^{n-1} E^{(r)}{}_b{}^p E^{(n-r)}{}_p{}^a e_a{}^m - \sum_{r=0}^{n-2} E^{(r)}{}_b{}^\mu E^{(n-r)}{}_\mu{}^a e_a{}^m . \end{aligned} \tag{6.11}$$

Note that the first line on the rhs is known to k-th order, because $E_p{}^a$ is so known by assumption and we have shown that $E_b{}^\mu$ can be computed to that order. The remaining terms are also known to k-th order, because they start at quadratic order in the fields. One argues in the same fashion for the remaining $E_\beta{}^M$ components of the inverse vielbein.

Let us now illustrate the use of the formulae above, by computing the vielbein exactly in θ and up to quadratic order in the fields.

6.1 Linear order

In this case $Q_{ab} = 0$ and formula (6.9) reduces to

$$T_{ab} = [\cosh\sqrt{\mathcal{O}}]_{ab}{}^{ef} T^{(0)}_{ef} + [\mathcal{O}^{-1/2}\sinh\sqrt{\mathcal{O}}]_{ab}{}^{ef} T^{(1)}_{ef} , \tag{6.12}$$

where the functions of $\mathcal{O}$ above are formally defined by their Taylor expansions around zero. Of course the series terminate at order θ^{32}. Note that in the linear approximation $T^{(1)}$ is given by

$$T^{(1)}_{ef} = \frac{1}{4}(\theta\Gamma^{cd})^\alpha R^{(0)}_{efcd\,\alpha} + 2(\theta\mathcal{T}_{[e}{}^{cdgh})^\alpha \delta_{f]}{}^m \partial_m G^{(0)}_{cdgh} . \tag{6.13}$$

Moreover substituting (6.12) into (4.1), we arrive at the following linear-order expression for the four-form

$$G_{abcd} = G^{(0)}_{abcd} + 6i\theta\Gamma_{[ab} \sum_{k=0} \left\{ \frac{1}{(2k+1)!}\mathcal{O}^k\ T^{(0)} + \frac{1}{(2k+2)!}\mathcal{O}^k\ T^{(1)} \right\}_{cd]} . \tag{6.14}$$

As a check the reader can verify for herself that

$$\theta^\mu\partial_\mu G_{abcd} = \theta^\alpha\nabla_\alpha G_{abcd} , \tag{6.15}$$

as follows from (6.14), (2.8), (6.12), taking into account that $\theta^\mu\partial_\mu\mathcal{O} = 2\mathcal{O}$ and $\theta^\mu\partial_\mu T^{(1)} = T^{(1)}$.

Equipped with (6.12), (6.14), we can now systematically derive expressions for all the other relevant superspace quantities. The strategy is straightforward: truncate to linear order the expressions for the superconnection and vielbein components given in section 4.2, substituting (6.12),(6.14) where necessary. For the superconnection components we get

$$\begin{aligned}\Omega_{\mu ab} &= \frac{i}{2}(\theta\mathcal{R}_{ab}{}^{cdef})_\mu G^{(0)}_{cdef}\\ &+\sum_{k=0}\frac{1}{2k+3}\Big(\frac{1}{2k+1}C_{1ab}{}^{ef}+\frac{1}{2}C_{2ab}{}^{ef}\Big)_{\mu\alpha}\left\{\frac{\mathcal{O}^k}{(2k)!}T^{(0)}\right\}_{ef}{}^{\alpha}+\\ &+\sum_{k=0}\frac{1}{2k+4}\Big(\frac{1}{2k+2}C_{1ab}{}^{ef}+\frac{1}{2}C_{2ab}{}^{ef}\Big)_{\mu\alpha}\left\{\frac{\mathcal{O}^k}{(2k+1)!}T^{(1)}\right\}_{ef}{}^{\alpha}\end{aligned} \tag{6.16}$$

and

$$\Omega_{mab}=\omega_{mab}+i\theta\mathcal{S}_{mab}{}^{ef}\sum_{k=0}\left\{\frac{\mathcal{O}^k}{(2k+1)!}T^{(0)}+\frac{\mathcal{O}^k}{(2k+2)!}T^{(1)}\right\}_{ef}. \tag{6.17}$$

For the vielbein components we get

$$E_\mu{}^\alpha=\delta_\mu^\alpha+\Delta E_\mu{}^\alpha\,, \tag{6.18}$$

where

$$\boxed{\begin{aligned}\Delta E_\mu{}^\alpha :=& \frac{i}{6}(D_1^{abcd})_\mu{}^\alpha G^{(0)}_{abcd}+\\ &+\sum_{k=0}\frac{1}{2k{+}4}\Big(\frac{F_1^{ef}}{(2k{+}3)(2k{+}1)}+\frac{F_2^{ef}}{2(2k{+}3)}+\frac{F_3^{ef}}{2(2k{+}1)}\Big)^\alpha{}_{\mu\beta}\left\{\frac{\mathcal{O}^k}{(2k)!}T^{(0)}\right\}_{ef}{}^\beta+\\ &+\sum_{k=0}\frac{1}{2k{+}5}\Big(\frac{F_1^{ef}}{(2k{+}4)(2k{+}2)}+\frac{F_2^{ef}}{2(2k{+}4)}+\frac{F_3^{ef}}{2(2k{+}2)}\Big)^\alpha{}_{\mu\beta}\left\{\frac{\mathcal{O}^k}{(2k+1)!}T^{(1)}\right\}_{ef}{}^\beta\end{aligned}} \tag{6.19}$$

and

$$E_m{}^\alpha=\Delta E_m{}^\alpha\,, \tag{6.20}$$

where

$$\boxed{\begin{aligned}\Delta E_m{}^\alpha :=& \Psi_m{}^\alpha+\frac{1}{4}(\theta\Gamma^{ab})^\alpha\omega_{mab}-(\theta\mathcal{T}_m{}^{abcd})^\alpha G^{(0)}_{abcd}+\\ &+i\sum_{k=0}\left\{\frac{\mathcal{O}^k}{(2k+2)!}T^{(0)}+\frac{\mathcal{O}^k}{(2k+3)!}T^{(1)}\right\}_{ef}{}^\beta(D_{2m}{}^{ef})_\beta{}^\alpha\,.\end{aligned}} \tag{6.21}$$

Finally, equations (4.6), (4.7) imply

$$E_\mu{}^a=-\frac{i}{2}(\Gamma^a\theta)_\mu+\Delta E_\mu{}^a\,, \tag{6.22}$$

where

$$
\boxed{\begin{aligned}
\Delta E_\mu{}^a &:= \frac{1}{24}(D_1^{bcde}\Gamma^a\theta)_\mu G^{(0)}_{bcde} - \\
&-\sum_{k=0} \frac{i(\Gamma^a\theta)_\alpha}{(2k+5)(2k+4)} \times \\
&\times \left(\frac{F_1^{ef}}{(2k+3)(2k+1)} + \frac{F_2^{ef}}{2(2k+3)} + \frac{F_3^{ef}}{2(2k+1)}\right)^\alpha{}_{\mu\beta} \left\{\frac{\mathcal{O}^k}{(2k)!}T^{(0)}\right\}_{ef}{}^\beta - \\
&- \sum_{k=0} \frac{i(\Gamma^a\theta)_\alpha}{(2k+6)(2k+5)} \times \\
&\times \left(\frac{F_1^{ef}}{(2k+4)(2k+2)} + \frac{F_2^{ef}}{2(2k+4)} + \frac{F_3^{ef}}{2(2k+2)}\right)^\alpha{}_{\mu\beta} \left\{\frac{\mathcal{O}^k}{(2k+1)!}T^{(1)}\right\}_{ef}{}^\beta
\end{aligned}} \tag{6.23}
$$

and

$$
E_m{}^a = \delta_m{}^a + \Delta E_m{}^a \,, \tag{6.24}
$$

where

$$
\boxed{\begin{aligned}
\Delta E_m{}^a &:= h_m{}^a - i(\Psi_m\Gamma^a\theta) - \frac{i}{8}(\theta\Gamma^{aef}\theta)\ \omega_{mef} + \frac{i}{2}(\theta\mathcal{T}_m{}^{bcde}\Gamma^a\theta)\ G^{(0)}_{bcde} + \\
&+ \sum_{k=0}\left\{\frac{\mathcal{O}^k}{(2k+3)!}T^{(0)} + \frac{\mathcal{O}^k}{(2k+4)!}T^{(1)}\right\}_{ef}{}^\beta (D_{2m}{}^{ef}\Gamma^a\theta)_\beta \,.
\end{aligned}} \tag{6.25}
$$

We will also need the inverse vielbein components $E_b{}^M$ to linear order. From equations (6.10), (6.11) we obtain

$$
\begin{aligned}
E_b{}^\mu &= -\Psi_b{}^\mu - \frac{1}{4}(\theta\Gamma^{ef})^\mu\omega_{bef} + (\theta\mathcal{T}_b{}^{cdef})^\mu G^{(0)}_{cdef} - \\
&- i\sum_{k=0}\left\{\frac{\mathcal{O}^k}{(2k+2)!}T^{(0)} + \frac{\mathcal{O}^k}{(2k+3)!}T^{(1)}\right\}_{ef}{}^\beta (D_{2b}{}^{ef})_\beta{}^\mu
\end{aligned} \tag{6.26}
$$

and

$$
\begin{aligned}
E_b{}^m &= e_b{}^m + \frac{i}{2}(\Psi_b\Gamma^m\theta) + \\
&+ \sum_{k=0}\left\{\frac{2k+1}{2}\frac{\mathcal{O}^k}{(2k+3)!}T^{(0)} + \frac{2k+2}{2}\frac{\mathcal{O}^k}{(2k+4)!}T^{(1)}\right\}_{ef}{}^\beta (D_{2b}{}^{ef}\Gamma^m\theta)_\beta \,.
\end{aligned} \tag{6.27}
$$

It is straightforward to check that to linear order the expressions above satisfy $E_b{}^M E_M{}^A = \delta_b{}^A$, as they should.

For the C-field we get

$$
C_{\mu\nu\sigma} = \frac{i}{8}(\Gamma^a\theta)_{(\mu}(\Gamma^b\theta)_\nu(\Gamma_{ab}\theta)_{\sigma)} + \Delta C_{\mu\nu\sigma} \,, \tag{6.28}
$$

where

$$\Delta C_{\mu\nu\sigma} := \sum_{n=0} \left\{ \frac{3i}{4(n+6)} (\Gamma^a\theta)_{(\mu} (\Gamma^b\theta)_\nu \Delta E^{(n)}{}_{\sigma)}{}^{\alpha} (\Gamma_{ab}\theta)_\alpha - \frac{3}{n+5} \Delta E^{(n)}{}_{(\mu}{}^{a} (\Gamma^b\theta)_\nu (\Gamma_{ab}\theta)_{\sigma)} \right\} . \tag{6.29}$$

$$C_{s\mu\nu} = \frac{1}{4} (\Gamma^a\theta)_{(\mu} (\Gamma_{ab}\theta)_{\nu)} \delta^b_s + \Delta C_{s\mu\nu} \,, \tag{6.30}$$

where

$$\begin{aligned} \Delta C_{s\mu\nu} := \sum_{n=0} \Bigg\{ & \frac{i}{4(n+5)} (\Gamma^a\theta)_\mu (\Gamma^b\theta)_\nu \Delta E^{(n)\alpha}_{s} (\Gamma_{ab}\theta)_\alpha - \frac{1}{n+4} \Delta E^{(n)a}_{s} (\Gamma^b\theta)_{(\mu} (\Gamma_{ab}\theta)_{\nu)} + \\ & + \frac{2i}{n+3} \Delta E^{(n)}{}_{(\mu}{}^{a} (\Gamma_{as}\theta)_{\nu)} + \frac{1}{n+4} \Delta E^{(n)}{}_{(\mu}{}^{\alpha} (\Gamma_{sa}\theta)_\alpha (\Gamma^a\theta)_{\nu)} \Bigg\} . \end{aligned} \tag{6.31}$$

$$C_{mn\sigma} = -\frac{i}{2} (\Gamma_{ab}\theta)_\sigma \delta^a_m \delta^b_n + \Delta C_{mn\sigma} \,, \tag{6.32}$$

where

$$\begin{aligned} \Delta C_{mn\sigma} := \sum_{n=0} \Bigg\{ & \frac{2i}{n+2} \Delta E^{(n)}{}_{[m}{}^{a} (\Gamma_{n]a}\theta)_\sigma - \frac{i}{n+2} \Delta E^{(n)\alpha}_{\sigma} (\Gamma_{mn}\theta)_\alpha \\ & - \frac{1}{n+3} \Delta E^{(n)}{}_{[m}{}^{\alpha} (\Gamma_{n]a}\theta)_\alpha (\Gamma^a\theta)_\sigma \Bigg\} . \end{aligned} \tag{6.33}$$

$$C_{mnp} = \Delta C_{mnp} \,, \tag{6.34}$$

where

$$\Delta C_{mnp} := C^{(0)}_{mnp} - \sum_{n=0} \frac{3i}{n+1} \Delta E^{(n)}{}_{[m}{}^{\alpha} (\Gamma_{np]}\theta)_\alpha \,. \tag{6.35}$$

6.2 Quadratic order and beyond.

In order to compute $T_{ab}{}^{\alpha}$ to quadratic order, we have to determine $Q_{ab}{}^{\alpha}$. Ignoring terms that start at cubic order, from equation (6.4) we have

$$\begin{aligned} Q^{(n)}{}_{ab}{}^{\alpha} \; = \; & \frac{i}{n(n-1)} (\mathcal{N}_{ab}{}^{c_1 \ldots c_6})^{\alpha}{}_{\beta} (G_{c_1 \ldots c_4} T_{c_5 c_6}{}^{\beta})^{(n-2)} - \\ & - \frac{i}{n(n-1)} (\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta} \; h_{|b]}{}^{m} \partial_m T^{(n-2)}{}_{ef}{}^{\beta} + \\ & + \frac{i}{n(n-1)} (\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta} \; \delta_{|b]}{}^{m} \sum_{r=0}^{n-2} (\Omega^{(r)}_m T^{(n-r-2)})_{ef}{}^{\beta} + \\ & + \frac{i}{n(n-1)} (\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta} \sum_{r=1}^{n-2} E^{(r)}{}_{|b]}{}^{m} \partial_m T^{(n-r-2)}{}_{ef}{}^{\beta} + \\ & + \frac{i}{n(n-1)} (\mathcal{M}_{[a|}{}^{ef})^{\alpha}{}_{\beta} \sum_{r=0}^{n-2} E^{(r)}{}_{|b]}{}^{\mu} \partial_\mu T^{(n-r-1)}{}_{ef}{}^{\beta} . \end{aligned} \tag{6.36}$$

Therefore $Q_{ab}{}^{\alpha}$ can be readily obtained by plugging into (6.36) the *linear* expressions for G, T, Ω_m, $E_b{}^M$ obtained in section 6.1. Then T is computed to quadratic order using equation (6.9) and G is computed to quadratic order through the recursion relation (4.1). The connection and the vielbein are obtained to quadratic order by using formulae (4.6), (4.7), (4.10), (4.12), (4.16), (4.19). Clearly, this procedure can be iterated to any desired order in the number of fields.

7. The supermembrane

The eleven-dimensional supermembrane can be described by a superembedding

$$Z : \Sigma^{(3|0)} \to M^{(11|32)} , \tag{7.1}$$

where

$$Z^{\underline{M}} := (X^{\underline{m}}, \theta^{\underline{\mu}}) \tag{7.2}$$

are the membrane supercoordinates. For the purposes of this section we pass to superembedding notation whereby all target-space (i.e. eleven-dimensional) indices are underlined. The world-volume theory of the membrane is given by [1, 2]

$$S = \int_\Sigma d\sigma^3 \{\sqrt{-g} + f^*C\} , \tag{7.3}$$

where

$$f^*C := \frac{1}{6}\varepsilon^{mnp}\partial_m Z^{\underline{P}}\partial_n Z^{\underline{N}}\partial_p Z^{\underline{M}} C_{\underline{MNP}} \tag{7.4}$$

is the pull back of the C-field onto the membrane world-volume and g is the determinant of the Green-Schwarz metric

$$g_{mn} := \left(\partial_m Z^{\underline{M}} E_{\underline{M}}{}^{\underline{a}}\right)\left(\partial_n Z^{\underline{N}} E_{\underline{N}}{}^{\underline{b}}\right)\eta_{\underline{ab}} . \tag{7.5}$$

7.1 Linear background

Using the results of section 6, we are now ready compute the linear coupling of the supermembrane to an on-shell eleven-dimensional background. For the metric (7.5) we find, to linear order,

$$g_{mn} = G_{mn} + \Delta g_{mn} , \tag{7.6}$$

where

$$G_{mn} := \Pi_m{}^{\underline{a}}\Pi_n{}^{\underline{b}}\eta_{\underline{ab}} , \tag{7.7}$$

$$\Delta g_{mn} := 2\Pi_{(m}{}^{\underline{a}}\partial_{n)} Z^{\underline{N}}\Delta E_{\underline{N}}{}^{\underline{b}}\,\eta_{\underline{ab}} \tag{7.8}$$

and

$$\Pi_m{}^{\underline{a}} := \partial_m X^{\underline{a}} - \frac{i}{2}(\partial_m\theta\Gamma^{\underline{a}}\theta) ; \qquad X^{\underline{a}} := X^{\underline{m}}\delta^{\underline{a}}_{\underline{m}} . \tag{7.9}$$

Note that G_{mn} is the Green-Schwarz metric for the case of flat target space. The linear-order components $(\Delta E_{\underline{M}}{}^{\underline{a}})$ of the eleven-dimensional vielbein were given explicitly in section 6.1, equations (6.19), (6.21), (6.23), (6.25). The determinant is given in the linear approximation by

$$\sqrt{-g} = \sqrt{-G}\,\left(1 + \Delta g_{mn}G^{mn}\right) , \tag{7.10}$$

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where G^{mn} is the inverse of G_{mn}. Moreover, for the Wess-Zumino term in (7.3) we have

$$f^*C = f^*\Delta C + \varepsilon^{mnp}\Big\{\frac{i}{4}\partial_m X^{\underline{a}}\partial_n X^{\underline{b}}(\partial_p\theta\Gamma_{\underline{ab}}\theta) + \\ + \frac{1}{8}\partial_m X^{\underline{a}}(\partial_n\theta\Gamma^{\underline{b}}\theta)(\partial_p\theta\Gamma_{\underline{ab}}\theta) - \frac{i}{48}(\partial_m\theta\Gamma^{\underline{a}}\theta)(\partial_n\theta\Gamma^{\underline{b}}\theta)(\partial_p\theta\Gamma_{\underline{ab}}\theta)\Big\} \tag{7.11}$$

where the linear-order components ($\Delta C_{\underline{MNP}}$) of the three-form potential were given explicitly in section 6.1, equations (6.29), (6.31), (6.33), (6.35).

To summarize, to linear order the coupling of the supermembrane to the eleven-dimensional background is given by

$$S = S_{flat} + \int_\Sigma d\sigma^3\{\sqrt{-G}G^{mn}\Delta g_{mn} + f^*\Delta C\}\,, \tag{7.12}$$

where

$$S_{\text{flat}} := \int_\Sigma d\sigma^3\Big\{\sqrt{-G} + \varepsilon^{mnp}\Big[\frac{i}{4}\partial_m X^{\underline{a}}\partial_n X^{\underline{b}}(\partial_p\theta\Gamma_{\underline{ab}}\theta) + \frac{1}{8}\partial_m X^{\underline{a}}(\partial_n\theta\Gamma^{\underline{b}}\theta)(\partial_p\theta\Gamma_{\underline{ab}}\theta) - \\ - \frac{i}{48}(\partial_m\theta\Gamma^{\underline{a}}\theta)(\partial_n\theta\Gamma^{\underline{b}}\theta)(\partial_p\theta\Gamma_{\underline{ab}}\theta)\Big]\Big\} \tag{7.13}$$

is the action of a supermembrane in flat eleven-dimensional target space.[6].

8. Outlook

Our result in section 7.1 can be used to read off the covariant membrane vertex operator, to linear order in the background fields. The covariant superstring and superparticle vertex operators in the linearized approximation, can also be obtained by reduction. One could also use the results of section 6.2 to go beyond the linear approximation, to any desired order in the number of background fields. This may be a promising approach, alternative to [15], to formulating covariant scattering amplitudes for the superstring or the superparticle.

By gauge-fixing to the light-cone gauge, one could of course compare with the existing results in the literature. The linearized light-cone membrane vertex operator of [13] terminates at order θ^5. It would be interesting to know if similar simplifications occur even when one introduces terms nonlinear in the background fields. In this case, one may hope that the full θ-expansion of the light-cone vertex operator, in a general supergravity background, may be computable.

As already mentioned, one could regularize (7.12) in order to make contact with the matrix model and compare with the partial results obtained in [12]. Moreover, our methods make it readily possible to go beyond the linear order in the background fields. Subtleties arise already at the quadratic order [9, 10] and it would be of interest to have an explicit matrix-model candidate in this case.

Another possible application concerns membrane (and also five-brane) instanton computations. These have so far been restricted to the case of membranes wrapping rigid,

[6] In order to translate the action (7.13) to the one given in [3, eq. (2.1)], see footnote 4.

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isolated supersymmetric cycles. In order to study the contribution of instantons with more zero modes, it is necessary to have knowledge of the higher terms in the θ-expansion of the supermembrane action. The explicit results of section 5 can be used for this purpose.

Finally let us note that the methods of this paper can be readily carried over to other superspaces in lower dimensions, as well as to the case of deformed eleven-dimensional supergravity, where the latter is obtained by modifying the torsion constraints as in [43, 44]. This modification will result in deforming the action of the supercovariant spinor derivative — and, therefore, the recursion relations (4.1) — by the inclusion of higher-order curvature terms. In its turn this will induce higher-order curvature corrections to the remaining recursion relations presented in section 4. In [43, 44] the deformations to the torsion constraints were parametrized in terms of certain superfields which were treated as 'black boxes'. In order to obtain explicit expressions however, these superfields would ultimately have to be expressed in terms of the fields in the massless multiplet. Unfortunately at present this remains a very difficult problem, although a systematic way to arrive at these explicit corrections has been proposed in [46].

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A. Definitions

For the convenience of the reader in this section we include an alphabetical index of various definitions used in this paper.

$(C_{1ab}{}^{ef})_{\alpha\beta}$, $(C_{2ab}{}^{ef})_{\alpha\beta}$: eqn (4.11)
$(D_1^{cdef})_\beta{}^\alpha$: eqn (4.17)
$(D_{2a}{}^{bc})_\beta{}^\alpha$: eqn (4.20)
$\mathcal{D}_m$: eqn (2.25)
ΔC_{MNP}: eqns (6.29),(6.31),(6.33),(6.35)
$\Delta E_M{}^A$: eqns (6.19),(6.21),(6.23),(6.25)
$e_m{}^a$: eqn (2.15)
$(F_1^{ef})^\alpha{}_{\beta\gamma}$, $(F_2^{ef})^\alpha{}_{\beta\gamma}$, $(F_3^{ef})^\alpha{}_{\beta\gamma}$: eqn (4.17)
Γ_m: eqn (2.22)
$h_m{}^a$: eqn (6.1)
$h_a{}^m$: eqn (6.5)
$(\mathcal{M}_a{}^{ef})^\alpha{}_\beta$: eqn (4.3)
$(\mathcal{N}_{ab}{}^{c_1\cdots c_6})^\alpha{}_\beta$: eqn (4.3)
$[\mathcal{O}]_{ab}{}^{ef}$: eqn (6.8)
$\omega_{mA}{}^B$: eqn (2.15)
$\Psi_m{}^\alpha$: eqn (2.15)
$\Psi_a{}^\mu$: eqn (3.16)

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$Q^{(n)}{}_{ab}{}^{\alpha}$: eqn (6.4)
$\mathcal{R}_{ab}{}^{cdef}$: eqn (2.6)
$S^{(n)}_{\{A\}}$: eqn (3.2)
$\mathcal{S}_{bcd}{}^{ef}$: eqn (2.6)
$\mathcal{T}_{a}{}^{bcde}$: eqn (2.6)
θ^{α}: eqn (3.14) .

B. Gamma matrices

In this section we explain our spinor notation and conventions for gamma matrices.

Spinor indices are denoted by lower-case Greek letters and spinors always carry the spinor index "upstairs". In eleven dimensions there is an antisymmetric charge-conjugation matrix,

$$C_{\alpha\beta} = -C_{\beta\alpha}\,, \tag{B.1}$$

which can be used to raise/lower indices on gamma matrices:

$$(\Gamma_a)_{\alpha\beta} := C_{\alpha\gamma}(\Gamma_a)^{\gamma}{}_{\beta}\,; \qquad (\Gamma_a)^{\alpha\beta} := C^{\alpha\gamma}(\Gamma_a)_{\gamma}{}^{\beta}\,, \tag{B.2}$$

where $C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$.

We denote by $\Gamma^{a_1\dots a_n}$ the antisymmetrized product of n gamma matrices

$$\Gamma^{a_1\dots a_n} := \Gamma^{[a_1}\dots\Gamma^{a_n]}\,. \tag{B.3}$$

For $n = 1, 2, 5$ the antisymmetrized products are symmetric in the spinor indices, whereas for $n = 0, 3, 4$ they are antisymmetric. In particular we have

$$\Gamma^{a}_{\alpha\beta} = \Gamma^{a}_{\beta\alpha}\,. \tag{B.4}$$

In our conventions the Majorana condition reads

$$\bar{\theta} = \theta^{Tr}C\,. \tag{B.5}$$

We can use the above to drop the bar on the fermionic coordinate θ. For example,

$$(\bar{\theta}\Gamma_a\lambda) := (\bar{\theta}\Gamma_a)_{\alpha}\lambda^{\alpha} = (\theta^{Tr}C)_{\beta}(\Gamma_a)^{\beta}{}_{\alpha}\lambda^{\alpha} = \theta^{\gamma}C_{\gamma\beta}(\Gamma_a)^{\beta}{}_{\alpha}\lambda^{\alpha} = \theta^{\gamma}(\Gamma_a)_{\gamma\alpha}\lambda^{\alpha} = (\theta\Gamma_a\lambda)\,. \tag{B.6}$$

C. Pure geometry

The expansions of section 5 simplify considerably in the case of a purely geometric background. By that we mean a bosonic background in which the four-form field strength vanishes.[7] Of course in this case the on-shell equations of motion impose the Ricci-flatness

[7] There is a potential global obstruction to the vanishing of the four-form [45]. We are ignoring such subtleties here.

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condition. Note that $C^{(0)}_{mnp}$ is not necessarily zero. This situation appears in many physically interesting settings, see for example [20].

In the case of a purely geometric background we have

$$T^{(1)}{}_{ab}{}^{\alpha} = \frac{1}{4}(\theta\Gamma^{cd})^{\alpha} R^{(0)}_{abcd} \tag{C.1}$$

and

$$\Psi_m{}^{\alpha} = G^{(0)}_{abcd} = T^{(0)}{}_{ab}{}^{\alpha} = T^{(2)}{}_{ab}{}^{\alpha} = 0\,. \tag{C.2}$$

Taking the above into account we find

$$\begin{aligned}
E^{(0)}{}_{\mu}{}^{a} &= 0\\
E^{(1)}{}_{\mu}{}^{a} &= -\frac{i}{2}(\Gamma^a\theta)_{\mu}\\
E^{(2)}{}_{\mu}{}^{a} &= 0\\
E^{(3)}{}_{\mu}{}^{a} &= 0\\
E^{(4)}{}_{\mu}{}^{a} &= 0\\
E^{(5)}{}_{\mu}{}^{a} &= \frac{i}{240}\left(F_1^{ef} + F_2^{ef} + 2F_3^{ef}\right)^{\alpha}{}_{\mu\beta}(\Gamma^a\theta)_{\alpha} T^{(1)\beta}_{ef}\,,\\
E^{(0)}{}_{m}{}^{a} &= e_m{}^{a}\\
E^{(1)}{}_{m}{}^{a} &= 0\\
E^{(2)}{}_{m}{}^{a} &= -\frac{i}{8}(\theta\Gamma^{aef}\theta)\omega_{mef}\\
E^{(3)}{}_{m}{}^{a} &= 0\\
E^{(4)}{}_{m}{}^{a} &= \frac{1}{24}(T^{(1)}_{ef} D_{2m}{}^{ef}\Gamma^a\theta)\\
E^{(5)}{}_{m}{}^{a} &= 0
\end{aligned} \tag{C.3, C.4}$$

and

$$\begin{aligned}
E^{(0)}{}_{\mu}{}^{\alpha} &= \delta_{\mu}{}^{\alpha}\\
E^{(1)}{}_{\mu}{}^{\alpha} &= 0\\
E^{(2)}{}_{\mu}{}^{\alpha} &= 0\\
E^{(3)}{}_{\mu}{}^{\alpha} &= 0\\
E^{(4)}{}_{\mu}{}^{\alpha} &= \frac{1}{40}\left(F_1^{ef} + F_2^{ef} + 2F_3^{ef}\right)^{\alpha}{}_{\mu\beta} T^{(1)\beta}_{ef}\,,\\
E^{(0)}{}_{m}{}^{\alpha} &= 0\\
E^{(1)}{}_{m}{}^{\alpha} &= \frac{1}{4}(\theta\Gamma^{ef})^{\alpha}\omega_{mef}\\
E^{(2)}{}_{m}{}^{\alpha} &= 0\\
E^{(3)}{}_{m}{}^{\alpha} &= \frac{i}{6}(T^{(1)}_{ef} D_{2m}{}^{ef})^{\alpha}
\end{aligned} \tag{C.5}$$

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$$E^{(4)}{}_m{}^\alpha = 0\,. \tag{C.6}$$

Expansions for the C field:

$$\begin{aligned}
C^{(0)}_{\mu\nu\sigma} &= 0\\
C^{(1)}_{\mu\nu\sigma} &= 0\\
C^{(2)}_{\mu\nu\sigma} &= 0\\
C^{(3)}_{\mu\nu\sigma} &= \frac{i}{8}(\theta\Gamma_{ab})_{(\mu}(\theta\Gamma^a)_\nu(\theta\Gamma^b)_{\sigma)}\\
C^{(4)}_{\mu\nu\sigma} &= 0\,,
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
C^{(0)}_{\mu\nu s} &= 0\\
C^{(1)}_{\mu\nu s} &= 0\\
C^{(2)}_{\mu\nu s} &= \frac{1}{4}(\theta\Gamma_{se})_{(\mu}(\theta\Gamma^e)_{\nu)}\\
C^{(3)}_{\mu\nu s} &= 0\\
C^{(4)}_{\mu\nu s} &= -\frac{i}{48}(\theta\Gamma^a)_\mu(\theta\Gamma^b)_\nu\left\{(\theta\theta)\omega_{sab} - \frac{1}{2}(\theta\Gamma_{abcd}\theta)\omega_s{}^{cd}\right\} +\\
&\quad + \frac{i}{48}(\theta\Gamma_{ab})_{(\mu}(\theta\Gamma^a)_{\nu)}(\theta\Gamma^{bef}\theta)\omega_{sef}\,,
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
C^{(0)}_{\sigma mn} &= 0\\
C^{(1)}_{\sigma mn} &= -\frac{i}{2}(\theta\Gamma_{mn})_\sigma\\
C^{(2)}_{\sigma mn} &= 0\\
C^{(3)}_{\sigma mn} &= \frac{1}{16}\omega_{[m|gh}(\theta\Gamma^{ghe}\theta)(\theta\Gamma_{|n]e})_\sigma +\\
&\quad + \frac{1}{8}\left\{(\theta\theta)\omega_{[mn]e} - \frac{1}{2}(\theta\Gamma^{gh}{}_{e[m}\theta)\omega_{n]gh}\right\}(\theta\Gamma^e)_\sigma\\
C^{(4)}_{\sigma mn} &= 0\,,
\end{aligned} \tag{C.9}$$

$$\begin{aligned}
\partial_{[s}C^{(0)}_{mnp]} &= 0\\
C^{(1)}_{mnp} &= 0\\
C^{(2)}_{mnp} &= \frac{3i}{4}\left\{(\theta\theta)\omega_{[mnp]} - \frac{1}{2}(\theta\Gamma^{gh}{}_{[mn}\theta)\omega_{p]gh}\right\}\\
C^{(3)}_{mnp} &= 0\\
C^{(4)}_{mnp} &= +\frac{1}{8}(T^{(1)}_{gh}D_{2[m}{}^{gh}\Gamma_{np]}\theta) +\\
&\quad + \frac{3}{32}(\theta\Gamma^{ghe}\theta)\omega_{[m|gh}\left\{(\theta\theta)\omega_{|np]e} - \frac{1}{2}(\theta\Gamma^{g'h'}{}_{e|n}\theta)\omega_{p]g'h'}\right\}\,.
\end{aligned} \tag{C.10}$$

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